The Expressive Power of Planar Flows

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Background: Normalizing Flows (NF)

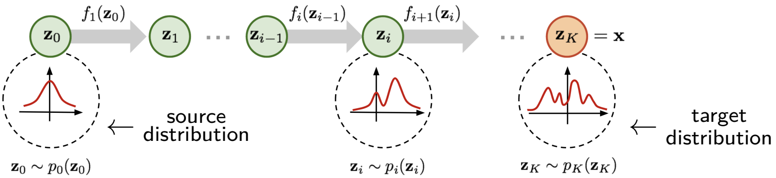

Figure: A NF model that transforms $p_0(z_0) \to p_K(z_K)$. [1]

- $p_K = f\#p_0$, where $f = f_K \circ \cdots \circ f_1$.
- Each f_i is simple, invertible, and parameterized.
- Solve MLE $\Rightarrow$ a generative model with computable likelihood.

[1] Figure by Lilian Weng, https://lilianweng.github.io/lil-log/assets/images/normalizing-flow.png

Problem Statement

Planar flows (Rezende and Mohamed, 2015):

$$f_{\mathsf{pf}}(z) = z + uh(w^\top z + b)$$

where $u, w, z \in \mathbb{R}^d, b \in \mathbb{R}$ with non-linearity $h : \mathbb{R} \to \mathbb{R}$.

- Setting: f composed of T planar flows.
  q – source(input) distribution, p – target distribution.
- Q$_1$–*Exact transformation*: when does it satisfy

$$p = f\#q \ (a.e.)$$

- Q$_2$–*Approximation*: given $\epsilon > 0$, is there a bound on T s.t.

$$\|f\#q - p\|_1 \leq \epsilon$$

Challenges – *Invertibility*

A NF is an *invertible* function!

Suppose $\mathcal{F}$ is a function class and $\mathcal{I} = \{\text{all invertible functions}\}$.

- $\mathcal{F}$ is a universal approximator $\not\Rightarrow$ $\mathcal{F} \cap \mathcal{I}$ can transform between arbitrary distributions.
- $\mathcal{F}$ has limited expressivity $\not\Rightarrow$ $\mathcal{F} \cap \mathcal{I}$ is not a universal approximator in transforming distributions.

Our technique: directly look at input-output distribution pairs.

Our Results

(i) 1-d: planar flows is a universal approximator in transforming distributions

- h can be arbitrary;
- $h = \text{ReLU}$ and $q = \mathcal{N}$.

(ii) High-d exact transformation: a topology matching condition

- h can be arbitrary $\Rightarrow \mathcal{N} \nrightarrow \mathcal{N}$;
- $h = \text{ReLU} \Rightarrow \text{MoG} \nrightarrow \text{MoG}, \text{Prod} \nrightarrow \text{Prod}$;
- compared to radial flows.

(iii) High-d approximation: a lower bound on T for local planar flows under certain conditions.

THANK YOU