CS292F Convex Optimization: Online (Projected Gradient Descent) **Spring 2020**

Lecture 14: May 13

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This lecture's notes illustrate some uses of various LaTeX macros. Take a look at this and imitate.

14.1 Online Convex Optimization

Review: Given

$$Regret_T(A) = sup_{\{f_1,\cdots,f_T\},f_i\in\mathbb{F}} \sum_{t=1}^{T} f_t(x_t) - \min_{u\in\mathbb{K}} \sum_{t=1}^{T} f_t(u)$$

where $\mathbb{F}$ is a family of convex functions, and $\mathbb{K}$ is a convex set.
Then, a setting of online convex optimization is:
Player declares algorithm A.
Adversary chooses $f_1, \cdots f_T \in \mathbb{F}$.
For $t = 1, \cdots, T$,

1. Player plays $x_t \in \mathbb{K}$, using A
2. Incur loss $f_t(x_t)$
3. Receive feedback

$$\begin{cases} \nabla f_t(x_t) \leftarrow \text{full information, feedback} \\ f_t(x_t) \leftarrow \text{bandit feedback} \\ f_t \text{ as function} \leftarrow \text{full function access} \end{cases}$$

14.2 Examples

Online linear models

$$x_t \in \mathbb{K} \subseteq \mathbb{R}^n, \phi_t \in \Phi \subseteq \mathbb{R}^n$$

$$f(x) = (< x, \phi_t > -\eta_t)^2 \text{ or } | < x, \phi_t > -\eta_t|, \text{ or } huber(| < x, \phi_t > -\eta_t|) \text{ or } logistic(< x, \phi_t > -\eta_t)$$

Online shortest path

$$\begin{aligned}
& x_t \in \mathbb{R}^{|E|} \geq 0 \text{ (probability of taking a particular edge)}, \\
\min_t \quad & < c, x_t > \\
s.t. \quad & \sum_{j:i=S,(i,j)\in E} x_t(i,j) = 1, \\
& \sum_{i:j=T,(i,j)\in E} x_t(i,j) = 1, \\
& \sum_{i:(i,j)\in E} = \sum_{j,(v,j)\in E} (v,j), \forall v \in E/\{S,T\}
\end{aligned}$$

Portfolio selection

Define the relative ratio for tock i as

$$r_t \in \mathbb{R}^n \geq 0.$$

Then,

$$\begin{aligned}
x_t &\in \Delta_n \\
\prod_{t=1}^{T} < r_t, x_t > &:= \text{ cumulative relative gain} \\
f_t(x_t) &= -\log(r_t^T x_t) \\
regret &= -\sum_{t=1}^{n} \log(r_t^T x_t) + \max_{x\in\Delta_n} \log(r_t^T x),
\end{aligned}$$

where $\log(r_t^T x_t)$ is the log of your cumulative return, and $\max_{x\in\Delta_n} \log(r_t^T x)$ is the rest fixed proportion constraint strategy in the hindsight.

14.3 Online Gradient Descent (OGD)

input: convex $\mathbb{K}$, horizon T, step size η_t, $t = 1, \cdots, T$
for $t = 1, \cdots, T$

- play x_t, observe $g_t \in \partial f_t(x_t)$
- update:
  - $x_{t+1/2} = x_t - \eta_t g_t$
  - $x_{t+1} = \prod_{\mathbb{K}}(x_{t+1/2})$
- Assumption:
  - G-Lipschitz: $f(x) - f(y) \leq G\|x - y\|_2$, $\forall x, y$ or $\|\nabla f(x)\|_2 \leq G$, $\forall x$
  - f convex
  - $diam(K) \leq D$, $\forall x, y \in K \implies \|x - y\|_2 \leq D$

Theorem 14.1. *Given above assumptions, the regret of OGD is bounded as*

$$Regret(u) := \sum_{t=1}^{T} f_t(x_t) - \sum_{t=1}^{T} f_t(u) \leq \frac{3}{2} GD\sqrt{T}, \forall u \in \mathbb{K}$$

when $\eta_t = \frac{D}{G\sqrt{t}}, t = 1, \cdots, T.$

Proof.

$$\begin{aligned}
f_t(x^*) &\geq f_t(x_t) + < g_t, x^* - x_t > \iff f_t(x_t) - f_t(x^*) \leq < g_t, x_t - x^* > \\
|x_{t+1} - x^*\|_2^2 &= \| \prod_{\mathbb{K}} (x_t - \eta_t g_t) - x^*\|_2^2 \\
&\leq \|x_t - \eta_t g_t - x^*\|_2^2 \\
&= \|x_t - x^*\|_2^2 + \eta_t^2 \|g_t\|^2 - 2\eta_t < g_t, x_t - x^* > \\
2(f_t(x_t) - f_t(x^*)) &\leq 2 < g_t, x_t - x^* > \leq \frac{\|x_t - x^*\|_2 - \|x_{t+1} - x^*\|^2}{\eta_t} + \eta_t \|g_t\|^2 \\
2Regret &= 2(\sum_t f_t(x_t) - \sum_t f_t(x^*)) \\
&\leq \sum_{t=1}^{T} \frac{1}{\eta_t} (\|x_t - x^*\|^2 - \|x_{t+1} - x^*\|^2) + (\sum_t \eta_t) \cdot G^2 \\
&= \sum_{t=1}^{T} (\frac{1}{\eta_t} - \frac{1}{\eta_{t-1}}) \cdot \|x_{t+1} - x^*\|_2^2 + (\sum_t \eta_t) \cdot G^2 \\
&\leq D^2 \sum_{t=1}^{T} (\frac{1}{\eta_t} - \frac{1}{\eta_{t-1}}) + (\sum_t \eta_t) \cdot G^2 \\
&= D^2 \frac{1}{\eta_T} + G^2 \sum_t \eta_t \\
&= DG\sqrt{T} + GD \sum_{t=1}^{T} \frac{1}{\sqrt{t}} \\
&\leq DG\sqrt{T} + 2DG\sqrt{T} \\
Regret &\leq \frac{3}{2} DG\sqrt{T}
\end{aligned}$$

given $\eta_0 = \infty, \eta_t = D/(G\sqrt{t})$. □

Strongly convex case f_t is m-strongly convex and G-lipschiz.

Theorem 14.2. *Choose* $\eta_t = \frac{1}{mt}$*, assume* f_t *is m-strongly convex. Then,*

$$Regret \leq \frac{G^2}{2m}(1 + \log(T)).$$

Proof.

$$
\begin{aligned}
2(f_t(x_t) - f_t(x^*)) &\leq 2\eta_t^T(x_t - x^*) - m\|x^* - x_t\|_2^2 && (1)\\
\|x_{t+1} - x^*\|_2^2 &= \|\prod_{\mathbb{K}}(x_t - \eta_t g_t) - x^*\|^2 \\
&\leq \|x_t - \eta_t g_t - x^*\|)2^2 \\
&= \|x_t - x^*\|^2 + \eta_t^2\|g_t\|^2 - 2\eta_t g_t^T(x_t - x^*) \\
2\eta_t^T(x_t - x^*) &\leq \frac{\|x_t - x^*\|^2 - \|x_{t+1} - x^*\|^2}{\eta_t} + \eta_t G^2 && (2)
\end{aligned}
$$

Combine (1), (2) to get

$$
\begin{aligned}
2(\sum_t f_t(x_t) - f_t(x^*)) &\leq \sum_t \frac{\|x_t - x^*\|^2 - \|x_{t+1} - x^*\|^2}{\eta_t} - m\|x^* - x_t\|^2 + G^2\sum_{t=1}^T \eta_t \\
&= \sum_{t=1}^T (\frac{1}{y_t} - \frac{1}{y_{t-1}} - m)\|x^* - x_t\|^2 - \frac{1}{y_T}\|x_{t+1} - x^*\| + G^2\sum_{t=1}^T \eta_t \\
&= \sum_{t=1}^T (mt - m(t-1) - m)\|x^* - x_t\|^2 + \frac{G^2}{m}(1 + \frac{1}{2}\cdots + \frac{1}{T}) \\
&\leq \frac{G^2}{m}(1 + \log(T)).
\end{aligned}
$$

□

14.4 Stochastic Subgradient Descent

Repeatedly update:

$$x_{t+1} = x_t + \eta_t \tilde{g}_t.$$

Assumption:

1. $\mathbb{E}[\tilde{g}_t|x_t] = g_t \in \partial f(x_t)$
2. $\mathbb{E}[\|\tilde(g)_t - E[\tilde{g}_t|x_t]\|^2|x_t] \leq G^2$
3. f is convex, and G-Lipschitz.

Under ϵ-suboptimality, $O(\frac{1}{\epsilon^2})$ iterations:

$$
\begin{aligned}
\mathbb{E}[f(x_k^*) - f^*] &\leq O(1/\sqrt{k}) \\
\mathbb{E}[f(\bar{x_k}) - f^*] &\leq O(1/\sqrt{k})
\end{aligned}
$$

Analysis:

$$
\begin{aligned}
f(\bar{x_T}) - f(x^*) &\leq \frac{1}{T}\sum_{t=1}^{T} f(x_t) - f(x^*) \\
&\leq \frac{1}{T}\sum_{t=1}^{T} < g_t, x_t - x^* > \\
&= \frac{1}{T}\sum_{t=1}^{T} < E[\tilde{g}_t|x_t], x_t - x^* > \\
&= \frac{1}{T}\sum_{t=1}^{T} E[< \tilde{g}_t, x_t - x^* > |x_t] \\
\mathbb{E}[f(\bar{x_T})] - f(x^*) &\leq \frac{1}{T}E[\sum_{t=1}^{T} < \tilde{g}_t, x_t > - \sum_{t=1}^{T} < \tilde{g}_t, X^* >] \\
&= \frac{1}{T}E[\sum_{t=1}^{T} f_t(x_t) - \sum_{t=1}^{T} f_t(x^*)] \\
&\leq Regret(OGD) \\
&= \frac{3}{2}\frac{GD\sqrt{T}}{t} \\
&= O(\frac{1}{\sqrt{T}})
\end{aligned}
$$

where $f_t(x) =< \tilde{g}_t, x >$ and x^* is the minimizer of f.