Analysis of a Damped Mass-Spring System with a Rubber Band

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14 October 2016

1 Introduction

This document works through four variants of harmonic motion with a mass and a spring. In the first case, the system contains only the mass and the spring. In the second case, damping is added. For the third case, the damping is removed and a rubber band is added parallel to the spring. For the fourth case, the damping is brought back and the system contains the spring, mass, rubber band, and damping.

The full equation for a mass and spring system with damping and a rubber band is:

\[ m \frac{d^2 y}{dt^2} + b \frac{dy}{dt} + k_1 y + k_2 h(y) = 10m, \]

where \( m \) is the mass pulling on the spring, \( b \) is the damping coefficient, \( k_1 \) is the spring constant of the spring, and \( k_2 \) is the "spring constant" related to the resistance created by the rubber band. \( h(y) \) is the piece-wise function describing the rubber band’s resistance, obeying Hooke’s Law, as it is stretched, and its passive state when the rubber band is under no tension. Our constants will be \( k_1 = 12.425 \) and \( k_2 = 4.7875 \). We will use a mass \( m = 1 \). The variable \( y \) stands for the displacement of the mass in relation to its equilibrium position. The variable \( dy/dt \) stands for the change in displacement in relation to the change in time, also known as velocity \( v \) which will be used later for simplicity’s sake in the graphical analysis. And the variable \( d^2 y/dt^2 \) is the change in velocity of the mass-spring system in relation to the change in time, in other words the acceleration of the system. In this document, MATLAB (using pplane8.m) and HPGSolver are used to determine the phase portraits and time graphs of these various systems.

2 Ideal Mass-Spring System with No Rubber Band

For the first section, \( b = 0, m = 1, k_2 = 0, \) and \( k_1 = 12.425 \), so the equation reads:

\[ \frac{d^2 y}{dt^2} + 12.425y = 10. \]

This equation describes a mass-spring system where there are no exterior damping factors and the rubber band does not exist. The phase portrait of this system is shown in Figure 1 along with the time graph of one of the solutions shown in Figure 2.
As evident in Figure 1, the solutions in the phase portrait are elliptical, thus indicating that in all initial conditions the solutions will be periodic and over time will return to its initial conditions. The initial conditions of the system are interpreted as the initial velocity \( v_0 \) and the initial displacement \( y_0 \) of the mass attached to the spring with which the system begins. Physically, this means that this system will continue to oscillate forever once the mass is set in motion, having no force acting against the movement. The system is at complete rest when the acceleration and velocity are set equal to zero. This equilibrium point is calculated at \( y = 0.8048 \).

From graphical analysis of Figure 2, the period \( T \) for this system is estimated to be 1.88 or \( 3\pi/5 \), meaning that this much time is required for the mass to return to its initial conditions.

### 3 Mass-Spring System with Damping but No Rubber Band

This section models a mass-spring system like the first section, except that it includes a damping factor on the mass. As before, \( m = 1 \), \( k_1 = 12.425 \), and \( k_2 = 0 \), but in this case, \( b = 1 \) and then \( b = 10 \). This system is then modeled by the equation:

\[
\frac{d^2y}{dt^2} + b\frac{dy}{dt} + 12.425y = 10.
\]

The phase portrait of this system is shown in Figure 3.

In analyzing the system with \( b = 1 \), it is found that the equilibrium of this system is at \( y = 0.8048 \). As evident in the phase portrait above, all solutions eventually converge to the equilibrium point, indicating...
that with damping the mass will oscillate to a resting point over time.

With \( b = 10 \), this system is modeled by the equation:

\[
\frac{d^2y}{dt^2} + 10 \frac{dy}{dt} + 12.425y = 10.
\]

The phase portrait of this system is shown in Figure 4.

![Figure 4: Phase portrait of the Damped Mass-Spring System but No Rubber Band (b=10)](image)

In analyzing the system with \( b = 10 \), it is found that the equilibrium of this system would be: \( y = 0.8048 \).

Compared to the system with \( b = 1 \), which oscillates to rest, the system with \( b = 10 \) goes directly to rest.

There is a value of \( b \) that separates the two instances above, one oscillating to rest (\( b = 1 \)) and the other going directly to rest (\( b = 10 \)). This bifurcation value is determined by numerical analysis with pplane8.m in MATLAB, resulting in a value of \( b \approx 5.8 \). When \( b \geq 5.8 \), solutions will not oscillate yet immediately go to rest, while when \( b < 5.8 \) solutions will oscillate.

4 Mass-Spring System with Rubber Band but No Damping

For this third section, \( b = 0, \ m = 1, \ k_1 = 12.425, \) and \( k_2 = 4.7875 \), so the equation reads:

\[
m \frac{d^2y}{dt^2} + 12.425y + 4.7875h(y) = 10m.
\]

This equation describes a system of a mass, spring, and rubber band with no exterior damping factors. The phase portrait of the system is shown in Figure 5, along with the time graphs of one of the solutions shown in Figure 6.
As evident in Figure 5, the solutions in this phase portrait are identical as the ones of the ideal mass-spring system with no damping (Ideal Mass-Spring System with No Rubber Band). This system’s solutions are also periodic, physically indicating that the mass will oscillate forever once set in motion. The only difference is that the system has a different equilibrium point. When the spring is compressed \((y < 0)\), it has the tendency to oscillate around equilibrium point \(y = 0.8045\). On the other hand, when it is stretched \((y \geq 0)\), the spring has the tendency to oscillate around equilibrium point: \(y = 0.5809\). These oscillations are due to the rubber band affecting the system only when \(y \geq 0\) (rubber band in tension). However, as shown in Figure 5, the equilibrium point of the system ends up at \(y = 0.5809\), which makes sense since the rubber band causes more initial tension, meaning that the initial displacement will be smaller.

As shown in Figure 6, the period \(T\) for this system is estimated to be 1.57 or \(\pi/2\), based on graphical analysis, meaning that this much time is required for the mass to return to its initial conditions.

5 Damped Mass-Spring System with Rubber Band

For this fifth section, \(b = 1\) and \(b = 10\), \(m = 1\), \(k_1 = 12.425\), and \(k_2 = 4.7875\), so the equation reads:

\[
\frac{d^2 y}{dt^2} + b \frac{dy}{dt} + 12.425y + 4.7875 \times h(y) = 10,
\]

This equation describes a system of a mass, spring, and rubber band with exterior damping factors \(b\). The phase portrait of the system is shown in Figure 7, along with the time graphs of one of the solutions shown in Figure 8.
In analyzing the system with $b = 1$, it is found that the equilibrium of this system is at $y = 0.8048$. This is the same as in Section 3. As evident in the phase portrait above, all solutions eventually converge to the equilibrium point, indicating that with damping the mass will oscillate to a resting point over time.

The two distinct behaviors shown in Figure 7 and Figure 8 indicate that the bifurcation value is between $b = 1$ and $b = 10$. Through numerical analysis, with the use of pplane8.m in MATLAB, we estimate the bifurcation value to be $b \approx 7.8$. This means that when $b < 7.8$, the spring will oscillate around its two equilibrium points and eventually come rest at point $y = 0.8045$, shown in Figure 7. When $b \geq 7.8$ the spring will not oscillate and go immediately to rest at point $y = 0.5809$, as shown in Figure 8.

Compared with the system behavior of Mass-Spring System with Damping but No Rubber Band, Figure 8 shows that with the rubber band the behavior of the system only differs when $y < 0$. By analyzing the equilibrium for both $y < 0$ and $y \geq 0$, we can see that, that, when the spring is compressed, it oscillates around a different equilibrium point ($y = 0.8045$) than when stretched ($y = 0.5809$). This is because the rubber band affects the system only when the spring is stretched where $y \geq 0$. 
6 Conclusion

This paper’s analysis of the various cases of the mass-spring system revealed some interesting behaviors relating a mass-spring system damped and tensioned with a rubber band. While the ideal system oscillates indefinitely, its phase portrait is periodic (Figure 1), the damped version comes to rest, its phase portrait spiraling towards equilibrium (Figure 3). This is because the damping reduces the mass’s movement in all directions, both when the spring is condensed and when it is compressed. It is important to note that the damping force has no effect on the placement of the equilibrium point. We also note that the path the mass takes toward equilibrium is dependent on the strength of the damping. The bifurcation value of the damping strength, where the mass’s path changes from an oscillation towards equilibrium to a direct descent to rest, changes when the rubber band is in use (Mass-Spring System with Rubber Band) as compared to the system without the rubber band (Mass-Spring System with Damping but No Rubber Band). This leads to the conclusion that the bifurcation point is based, not on the damping in the system, but on the other system factors, which in this case are the spring and the rubber band. It can be assumed that the bifurcation point would change if $k_1$ or $k_2$ changed, but that is not an observed behavior since $k_1$ and $k_2$ never change in this model.

The pattern of the mass’s oscillation can be broadly predicted based on which factors are present and the level of damping. If there is neither damping nor a rubber band on the mass-spring system, the mass will oscillate indefinitely. If there is damping, and the damping is less than the critical damping point, the mass will oscillate for a while, but eventually reach equilibrium and rest. If the damping is greater than the bifurcation value, the mass will move almost directly to rest and equilibrium without oscillation. If there is no damping but there is a rubber band, the mass will oscillate indefinitely, much like what happens in the ideal system. The rubber band increases the overall effective spring constant, although only in one direction. This case results in a similarly shaped path for the mass, but a different equilibrium point.

If there is both damping and a rubber band, the previous two effects are combined. The mass will oscillate to rest when the damping force is less than the bifurcation point, and will move directly to rest when the force is greater than the bifurcation value. The equilibrium point here also depends on the value of $y$, and will be the same as the two different equilibrium points found in the damping without a rubber band system and in the rubber band without damping system.