(co)induction: it's a thing!

Valentin Robert
University of California, San Diego

INDUCTION

Natural number induction

Prove a property for all natural numbers:

$$\forall\ n \in \mathbb{N},\ P(n)$$

Proof method:

- **prove `P(0)`**
- **prove `P(n+1)` while assuming `P(n)`**

Structural induction

Prove a property for all lists:

$$\forall\ l \in [T],\ P\ l$$

$$\frac{}{\texttt{nil} \in [T]} \qquad \frac{h \in T \quad t \in [T]}{h::t \in [T]}$$

Proof method:

- prove `P nil`
- prove `P (h::t)` while assuming `P t`

Structural induction

Natural number induction is structural

$$\frac{}{0 \in \mathbb{N}} \qquad \frac{n \in \mathbb{N}}{S\ n \in \mathbb{N}}$$

Proof method:

- prove P `0`
- prove P `(S n)` while assuming P `n`

What is an inductive type?

Given a set of introduction rules, to be read forward, a type is inductively defined, equivalently, as:

- **the set of objects built by finite proofs over these rules**
- **the smallest set closed under these rules**
- **the least fixed point of the underlying endofunctor**

What is an inductive type?

$$\frac{}{\mathbf{0} \in \mathbb{N}} \qquad \frac{\mathbf{n} \in \mathbb{N}}{\mathbf{S}\ \mathbf{n} \in \mathbb{N}}$$

- **the set of objects built by finite proofs over these rules**

$$\frac{}{\mathbf{0} \in \mathbb{N}}$$

$\mathbf{0}$

$$\cfrac{\cfrac{\cfrac{}{\mathbf{0} \in \mathbb{N}}}{\mathbf{S}\ \mathbf{0} \in \mathbb{N}}}{\mathbf{S}\ (\mathbf{S}\ \mathbf{0}) \in \mathbb{N}}$$

$\mathbf{2}$

What is an inductive type?

$$\frac{e \equiv 0}{e \in \mathbb{N}} \qquad \frac{n \in \mathbb{N} \quad e \equiv S\ n}{e \in \mathbb{N}}$$

- the smallest set closed under these rules
- the least fixed point of the underlying endofunctor
  - `F : Set → Set`
  - `F(X) ≜ X`
    `∪ {e | e ≡ 0}`
    `∪ {e | n ∈ X, e ≡ S n}`

What is an inductive type?

If an element belongs in an inductive type, this means that it satisfies some structural properties.

It has been constructed using a finite number of its rules (constructors).

Induction proof method

In order to prove that $\texttt{t} \in \texttt{P}$

Prove that $\texttt{t} \in \texttt{T}$

And prove that P is closed under T’s rules

Why is it sound?

A proof-theoretic argument:

P is true for some element `t` if we can build a finite proof term of `t ∈ P`.

To build this proof, follow `t`'s structure, applying the corresponding rules.

Why is it sound?

A set-theoretic argument:

The set T is the smallest set closed under its rules. Therefore, proving that P is closed under the same rules implies that T $\subseteq$ P.

Example: call-by-name reduction

$$\frac{}{\texttt{λx.e} \Downarrow \texttt{λx.e}}$$

$$\frac{\texttt{e1} \Downarrow \texttt{λx.b} \qquad \texttt{b[e2/x]} \Downarrow \texttt{e'}}{\texttt{e1 e2} \Downarrow \texttt{e'}}$$

Example: call-by-name reduction

$$\frac{\texttt{a} \equiv \lambda\texttt{x.e} \qquad \texttt{b} \equiv \lambda\texttt{x.e} \qquad \texttt{z} \equiv \texttt{(a, b)}}{\texttt{z} \in \Downarrow}$$

$$\frac{\texttt{(e1, }\lambda\texttt{x.b)}\in\Downarrow \qquad \texttt{(b[e2/x], e')}\in\Downarrow \qquad \texttt{z}\equiv\texttt{(e1 e2, e')}}{\texttt{z} \in \Downarrow}$$

COINDUCTION

What is a coinductive type?

"COINDUCTIVE TYPES MODEL INFINITE STRUCTURES UNFOLDED ON DEMAND, LIKE POLITICIANS' EXCUSES"

- CONOR MCBRIDE

What is a coinductive type?

Given a set of introduction rules, to be read backward, a type is coinductively defined, equivalently, as:

- the set of objects built by finite or infinite proofs over these rules
- the largest set closed under these rules
- the greatest fixed point of the underlying endofunctor

What is a coinductive type?

If an element belongs in a coinductive type, this means that it satisfies some observational properties.

It can be destructed using a finite number of its rules (destructors).

Coinductive lists

$$\frac{}{\texttt{nil} \in [\texttt{T}]} \qquad \frac{\texttt{h} \in \texttt{T} \quad \texttt{t} \in [\texttt{T}]}{\texttt{h::t} \in [\texttt{T}]}$$

Any object that can be observed:

- **to be `nil`**
- **to be `h::t` where `h ∈ T` and `t` can (corecursively) be observed in that way**

Coinductive streams

$$\frac{\texttt{h} \in \texttt{T} \qquad \texttt{t} \in \texttt{[T]}}{\texttt{h::t} \in \texttt{[T]}}$$

Any object that can be observed:

- **to be `h::t` where h `∈ T` and `t` can (corecursively) be observed in that way**

Coinduction proof method

In order to prove that `t` ∈ `T`

Prove that `t` ∈ `P`

And prove that P is closed under T's rules

Why is it sound?

A proof-theoretic argument:

`t ∈ T` if we can build a finite or infinite proof term.

No matter how far we observe `t` to be in P, we can prove that the observation so far is in T using the corresponding rule.

Why is it sound?

A set-theoretic argument:

The set T is the largest set closed under its rules. Therefore, proving that P is closed under the same rules implies that P $\subseteq$ T.

Coinduction proof method

In order to prove that `t` ∈ `T`

Prove that `t` ∈ `P`

And prove that P is closed under T's rules

But… how do we prove `t` ∈ `P`!?

Well… don't define P coinductively ☺

What is a coinductive type?

$$\frac{}{0 \in \mathbb{N}} \qquad \frac{n \in \mathbb{N}}{S\ n \in \mathbb{N}}$$

Largest set closed under these rules:

$\{ 0,\ S\ 0,\ S\ (S\ 0),\ \dots \} \cup \{\omega\}$

One of these is infinite

What is a coinductive type?

$$\frac{e \equiv 0}{e \in \mathbb{N}} \qquad \frac{n \in \mathbb{N} \quad e \equiv S\ n}{e \in \mathbb{N}}$$

Largest set closed under these rules:

`{ 0, S 0, S (S 0), … } ∪ {ω}`

One of these is infinite

Example: call-by-name divergence

$$\frac{\texttt{e1}\Uparrow}{\texttt{(e1 e2)}\Uparrow}$$

$$\frac{\texttt{e1} \Downarrow \lambda\texttt{x.b} \qquad \texttt{b[e2/x]}\Uparrow}{\texttt{(e1 e2)}\Uparrow}$$

EQUALITY

Judgmental equality

$\equiv$ is the least congruence including

- α safe renaming of bound variables
- β reduction of applied λ-terms
- δ unfolding of definitions
- ζ reduction of let-bindings
- η $f \equiv \lambda x.f\ x$
- ι reduction of:
  - pattern-matching over known constructs
  - fixes over known producers
  - cofixes under known consumers

Propositional equality

Problem:

$$1 + x \not\equiv x + 1$$

Solution:

$$\frac{x \in T}{x = x}$$

Reminder:

$$\frac{x \in T \qquad y \in T \qquad x \equiv y}{(T,\ x,\ y) \in =}$$

Propositional equality

$$
\dfrac{\dfrac{}{\texttt{1+0} \equiv \texttt{0+1}}}{\texttt{1+0 = 0+1}}
\qquad
\dfrac{\dfrac{\dfrac{\dfrac{}{\texttt{S (1 + n)} \equiv \texttt{S (1 + n)}}}{\texttt{S (1 + n) = S (1 + n)}}}{\forall \texttt{n, 1+n = n+1} \to \texttt{S (1 + n) = S (n + 1)}}}{\forall \texttt{n, 1+n = n+1} \to \texttt{1 + S n = S n + 1}}
$$

$$
\overline{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}
$$

$$
\texttt{1 + x = x + 1}
$$

More values, more problems

```
comap (+1) zeroes ≢ ones
comap (+1) zeroes ≠ ones
```

A coprogram is neither judgmentally nor propositionally equal to its unfolding.

Would make judgmental equality undecidable…

Also, coprograms break subject reduction…

Congruence

Equivalence relation respecting the constructors

$$\frac{}{\texttt{x = x}}$$

$$\frac{\texttt{e1 = e2} \qquad \texttt{e1' = e2'}}{\texttt{e1 e1' = e2 e2'}}$$

$$\frac{\texttt{e1 = e2}}{\texttt{λx. e1 = λx. e2}}$$

Bisimulation equivalence

Equivalence relation respecting the destructors

$$\frac{\texttt{e1}\Uparrow \qquad \texttt{e2}\Uparrow}{\texttt{e1} \approx \texttt{e2}}$$

$$\frac{\texttt{e1}\Downarrow\lambda\texttt{x.b1} \qquad \texttt{e2}\Downarrow\lambda\texttt{x.b2} \qquad \forall\texttt{e},\ \texttt{b1[e/x]} \approx \texttt{b2[e/x]}}{\texttt{e1} \approx \texttt{e2}}$$

$$\frac{\forall\sigma,\ \texttt{e1}[\sigma] \approx \texttt{e2}[\sigma]}{\texttt{e1} \approx \texttt{e2}}$$