To Do

- Homework 2 (Direct Lighting) due Apr 25
- Assignment is on UCSD Online
- START EARLY (NOW)

Motivation

Rendering = integration
- Reflectance equation: Integrate over incident illumination
- Rendering equation: Integral equation

Many sophisticated shading effects involve integrals
- Antialiasing
- Soft shadows
- Indirect illumination
- Caustics

Example: Soft Shadows

\[ E(x) = \frac{1}{|\Omega|} \int_{\Omega} L(x, \omega) \cos \theta \, d\omega \]

Challenges
- Visibility and blockers
- Varying light distribution
- Complex source geometry

Source: Agrawala, Ramamoorthi, Heirich, Moll, 2000

Monte Carlo

- Algorithms based on statistical sampling and random numbers
- Coined in the beginning of 1940s. Originally used for neutron transport, nuclear simulations
  - Von Neumann, Ulam, Metropolis, …
- Canonical example: 1D integral done numerically
  - Choose a set of random points to evaluate function, and then average (expectation or statistical average)

Monte Carlo Algorithms

Advantages
- Robust for complex integrals in computer graphics (irregular domains, shadow discontinuities and so on)
- Efficient for high dimensional integrals (common in graphics: time, light source directions, and so on)
- Quite simple to implement
- Work for general scenes, surfaces
- Easy to reason about (but care taken re statistical bias)

Disadvantages
- Noisy
- Slow (many samples needed for convergence)
- Not used if alternative analytic approaches exist (but those are rare)
Outline
- Motivation
- Overview, 1D integration
- Basic probability and sampling
- Monte Carlo estimation of integrals

Integration in 1D
\[ \int_{a}^{b} f(x) \, dx =? \]

We can approximate
\[ \int_{a}^{b} f(x) \, dx = \frac{1}{N} \sum_{i=1}^{N} f(x_i) \]
Standard integration methods like trapezoidal rule and Simpson's rule
Advantages:
- Converges fast for smooth integrands
- Deterministic
Disadvantages:
- Exponential complexity in many dimensions
- Not rapid convergence for discontinuities

Or we can average
\[ \int_{a}^{b} f(x) \, dx = \mathbb{E}(f(x)) \]

Estimating the average
\[ \int_{x_{i-1}}^{x_i} f(x) \, dx = \frac{1}{N} \sum_{i=1}^{N} f(x_i) \]
Monte Carlo methods (randomly choose samples)
Advantages:
- Robust for discontinuities
- Converges reasonably for large dimensions
- Can handle complex geometry, integrals
- Relatively simple implement, reason about

Other Domains
\[ \int_{a}^{b} f(x) \, dx = b - a \sum_{i=1}^{N} f(x_i) \]
Multidimensional Domains

Same ideas apply for integration over …
- Pixel areas
- Surfaces
- Projected areas
- Directions
- Camera apertures
- Time
- Paths

\[ \int f(x) \, dx = \frac{1}{N} \sum_{i=1}^{N} f(x_i) \]

Outline

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Random Variables

- Describes possible outcomes of an experiment
- In discrete case, e.g. value of a dice roll \( x = 1-6 \)
- Probability \( p \) associated with each \( x \) (1/6 for dice)
- Continuous case is obvious extension

Expected Value

- Expectation
  - Discrete: \( E(x) = \sum_{i=1}^{N} p \cdot x_i \)
  - Continuous: \( E(x) = \int p(x) f(x) \, dx \)

  For Dice example:
  \[
  E(x) = \frac{1}{6} \sum_{i=1}^{6} x_i = \frac{1}{6} (1 + 2 + 3 + 4 + 5 + 6) = 3.5
  \]

Continuous Probability Distributions

<table>
<thead>
<tr>
<th>PDF ( p(x) )</th>
<th>Uniform</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p(x) \geq 0 )</td>
<td>[ \int_{0}^{1} p(x) , dx = 1 ]</td>
</tr>
</tbody>
</table>

CDF \( P(x) \)

- \( P(x) = \Pr(X < x) \)
- \( P(1) = 1 \)
- \( \Pr(\alpha \leq X \leq \beta) = \int_{\alpha}^{\beta} p(x) \, dx \)
- \( = P(\beta) - P(\alpha) \)

Sampling Techniques

Problem: how do we generate random points/directions during path tracing?
- Non-rectilinear domains
- Importance (BRDF)
- Stratified
Generating Random Points

Uniform distribution:
- Use random number generator

Specific probability distribution:
- Function inversion
- Rejection
- Metropolis

Common Operations

Want to sample probability distributions
- Draw samples distributed according to probability
- Useful for integration, picking important regions, etc.

Common distributions
- Disk or circle
- Uniform
- Upper hemisphere for visibility
- Area luminaire
- Complex lighting like an environment map
- Complex reflectance like a BRDF

Generating Random Points

Cumulative probability distribution function

\[ P(x) = \Pr(X < x) \]

Construction of samples

Solve for \[ X = P^{-1}(U) \]

Must know:
1. The integral of \( p(x) \)
2. The inverse function \( P^{-1}(x) \)

Example: Power Function

Assume
\[ p(x) = (n+1)x^n \]
\[ P(x) = x^{n+1} \]
\[ X \sim p(x) \implies X = P^{-1}(U) = \sqrt[n+1]{U} \]

Trick
\[ Y = \max(U_1, U_2, \ldots, U_n) \]
\[ \Pr(Y < x) = \prod_{i=1}^{n} \Pr(U_i < x) = x^{n+1} \]
**Sampling a Circle**

\[
A = \frac{3}{2} \int_0^1 r^2 dr - \frac{1}{4} \int_0^1 r^2 dr - \frac{1}{2} \int_0^1 \frac{r}{2} dr = \pi
\]

\[
p(r, \theta) = p(r) p(\theta)
\]

\[
p(\theta) = \frac{1}{2\pi} \quad \theta = 2\pi U_1
\]

\[
p(r) = 2r \\
p(r) = r^2
\]

\[
r = \sqrt{U_2}
\]

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**Rejection Sampling**

\[
I = \int f(x) \, dx = \int_y dx dy
\]

Algorithm:

- Pick \( U_1 \) and \( U_2 \)
- Accept \( U_1 \) if \( U_2 < f(U_1) \)

Wasteful? Efficiency = Area × Area of rectangle

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**Sampling a Circle: Rejection**

\[
\text{do } \{
X = 1 - 2U_1 \\
Y = 1 - 2U_2 \\
\text{while( } X^2 + Y^2 > 1 \text{ )}
\]

May be used to pick random 2D directions

Circle techniques may also be applied to the sphere

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**Outline**

- Motivation
- Overview, 1D integration
- Basic probability and sampling
- Monte Carlo estimation of integrals
Monte Carlo Path Tracing

Big diffuse light source, 20 minutes
Motivation for rendering in graphics: Covered in detail in next lecture

1000 paths/pixel

Monte Carlo Path Tracing

Estimating the average

$$\int_0^1 f(x) \, dx = \frac{1}{N} \sum_{i=1}^{N} f(x_i)$$

Monte Carlo methods (randomly choose samples)

Advantages:
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Monte Carlo Integration

Definite integral

$$I(f) = \int_0^1 f(x) \, dx$$

Expectation of $$f$$

$$E[f] = \int_0^1 f(x) \, p(x) \, dx$$

Random variables

$$X_i \sim p(x)$$

$$Y_i = f(X_i)$$

Estimator

$$F_N = \frac{1}{N} \sum_{i=1}^{N} Y_i$$

Unbiased Estimator

$$E[F_N] = I(f)$$

$$E[F_N] = \frac{1}{N} \sum_{i=1}^{N} Y_i$$

Properties

$$E[\sum Y_i] = \sum E[Y_i]$$

$$E[aY] = aE[Y]$$

Direct Lighting – Directional Sampling

$$E(x) = \int_{\Omega} L(x, \omega) \cos \theta \, d\omega$$

Ray intersection $$x'(x, \omega)$$

Sample $$\omega$$ uniformly by $$\Omega$$

$$Y_i = L(x'(x, \omega), -\omega) \cos \theta \, 2 \pi$$
Variance for Dice Example?

- Work out on board (variance for single dice roll)

Variance decreases as 1/N
Error decreases as 1/\sqrt{N}
**Variance Reduction**

Efficiency measure

\[ \text{Efficiency} \propto \frac{1}{\text{Variance} \cdot \text{Cost}} \]

Techniques
- Importance sampling
- Stratified sampling, ...

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**Variance Reduction Techniques**

- Importance sampling
- Stratified sampling

\[ \int_{\Omega} f(x) dx \approx \frac{1}{N} \sum_{i=1}^{N} f(x_i) \]

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**Importance Sampling**

Put more samples where \( f(x) \) is bigger

\[ E[f(x)] = \frac{1}{N} \sum_{i=1}^{N} Y_i \]

\[ Y_i = \frac{f(x_i)}{p(x_i)} \]

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**Importance Sampling**

This is still unbiased

\[ E[Y_i] = \int_{\Omega} Y(x)p(x) dx \]

\[ = \int_{\Omega} f(x) dx \]

for all \( N \)

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**Importance Sampling**

- Zero variance if \( p(x) \sim f(x) \)

\[ p(x) = cf(x) \]

\[ Y_i = \frac{f(x_i)}{p(x_i)} = \frac{1}{c} \]

\[ \text{Var}(Y) = 0 \]

Less variance with better importance sampling

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**Stratified Sampling**

- Estimate subdomains separately

\[ E[f(x)] = \frac{1}{N} \sum_{i=1}^{N} Y_i \]

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Stratified Sampling

- This is still unbiased

\[ F_N = \frac{1}{N} \sum_{i=1}^{N} f(x_i) \]

\[ = \frac{1}{N} \sum_{i=1}^{N} N_i f_i \]

\[ \mathbb{E}(f(x)) \]

Stratified Sampling

- Less overall variance if less variance in subdomains

\[ \text{Var}[F_N] = \frac{1}{N^2} \sum_{i=1}^{N} N_i \text{Var}[f_i] \]

\[ \mathbb{E}(f(x)) \]

More Information

- Veach PhD thesis chapter (linked to from website)

- Course Notes (links from website)