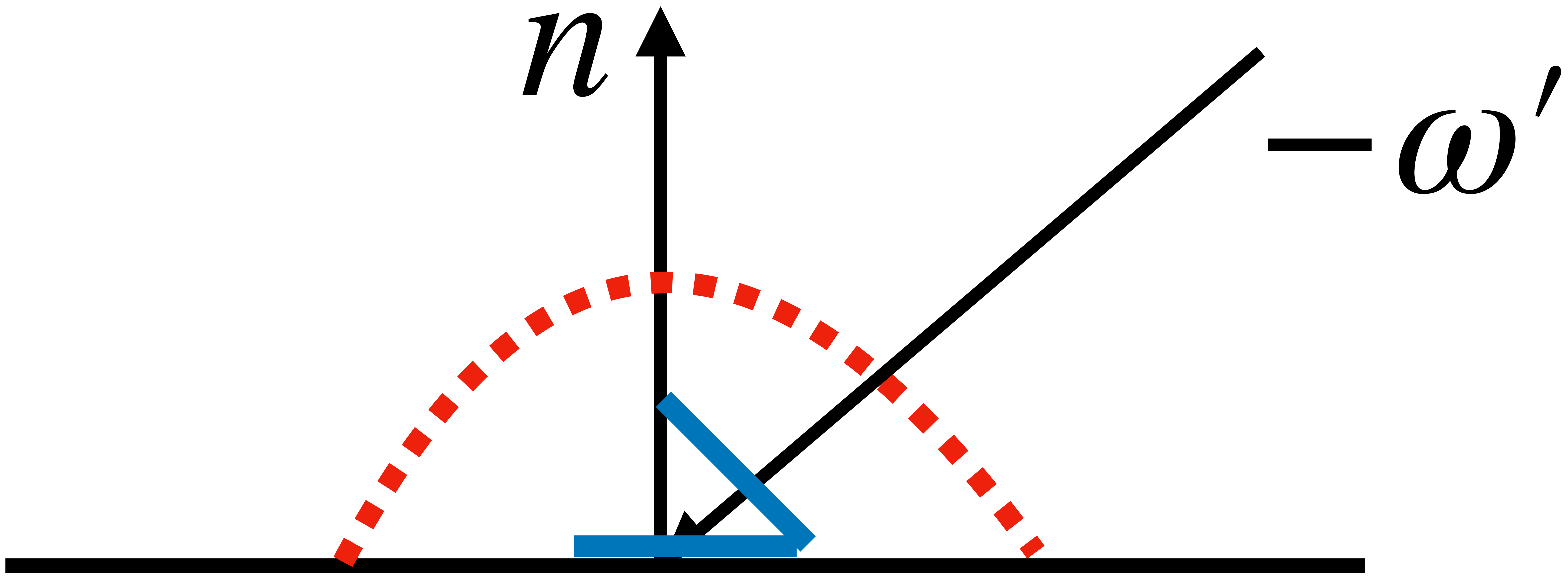
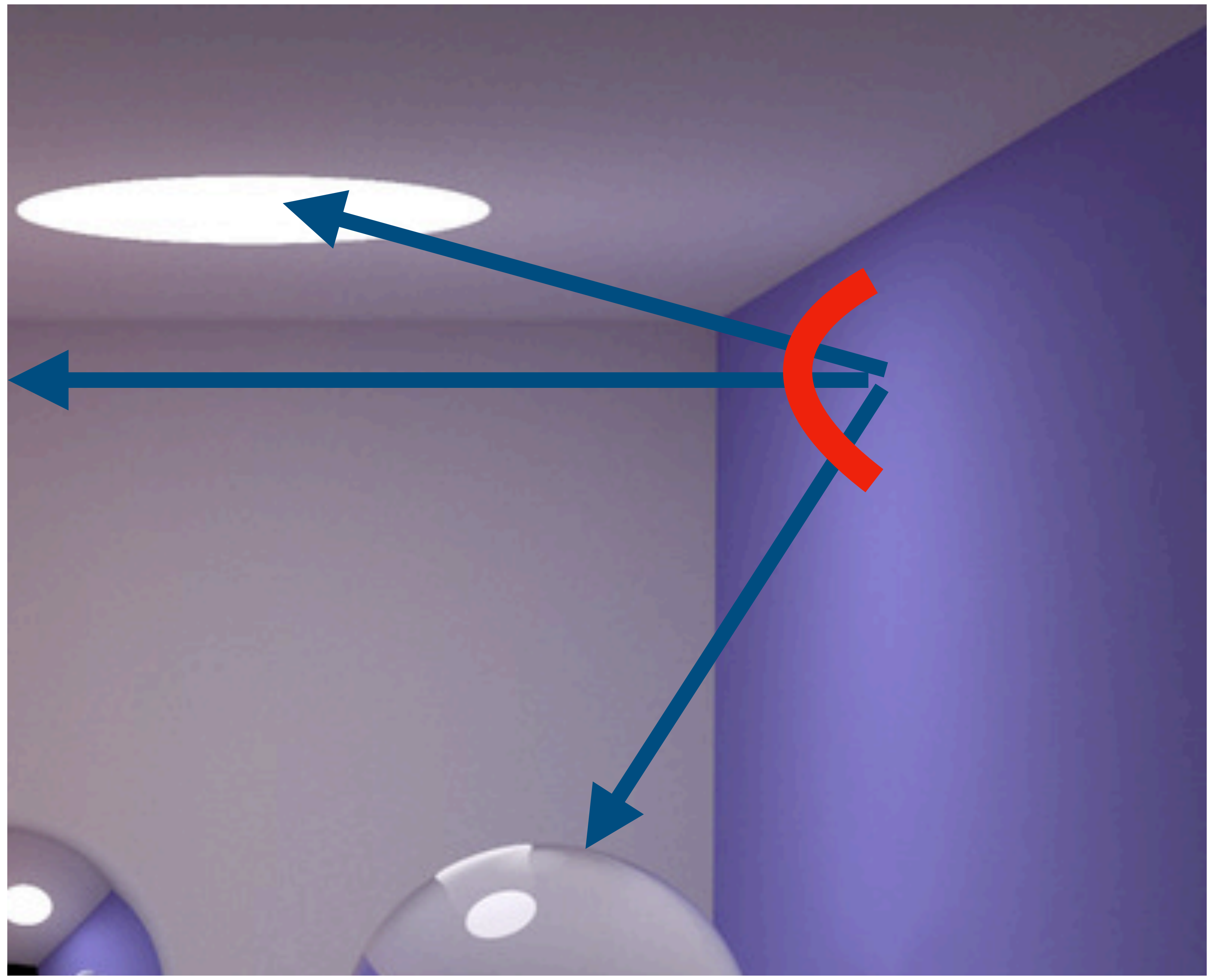
Goal: sample this hemispherical integral

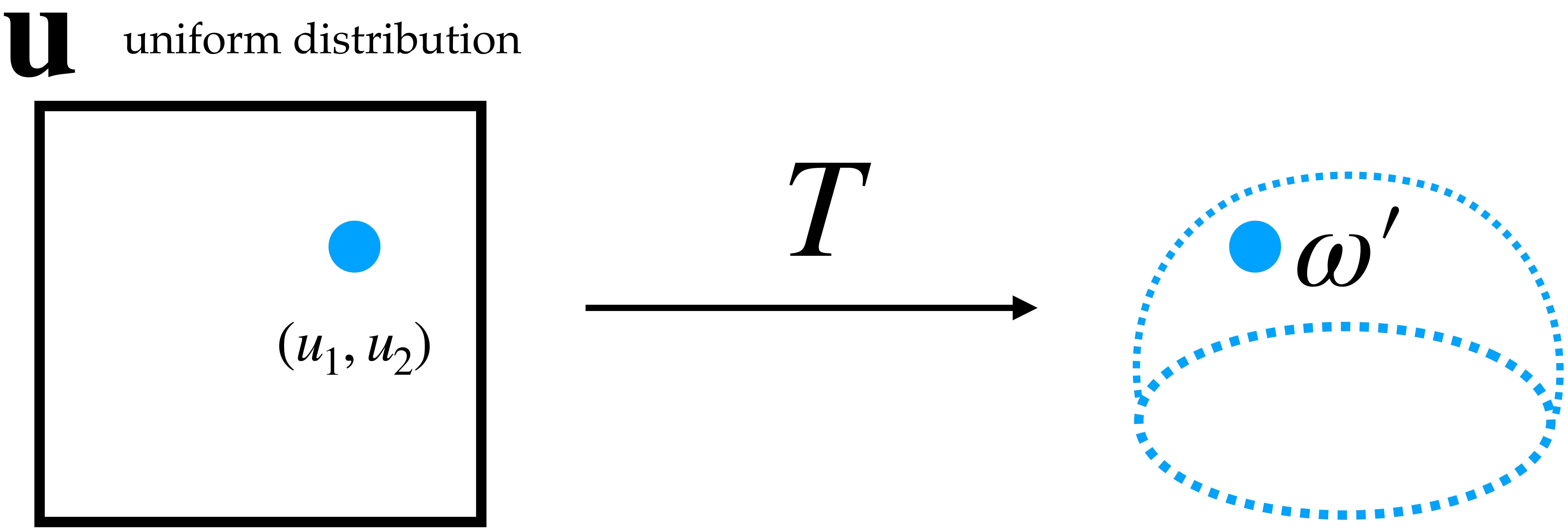
$$\iint \frac{\text{reflectance}}{\pi} L(\omega')\,|\,\omega' \cdot n\,|\,\mathrm{d}\omega' = \iint L'(\omega')\,|\,\omega' \cdot n\,|\,\mathrm{d}\omega'$$





the cosine term is the ratio between an area on hemisphere and an area on surface

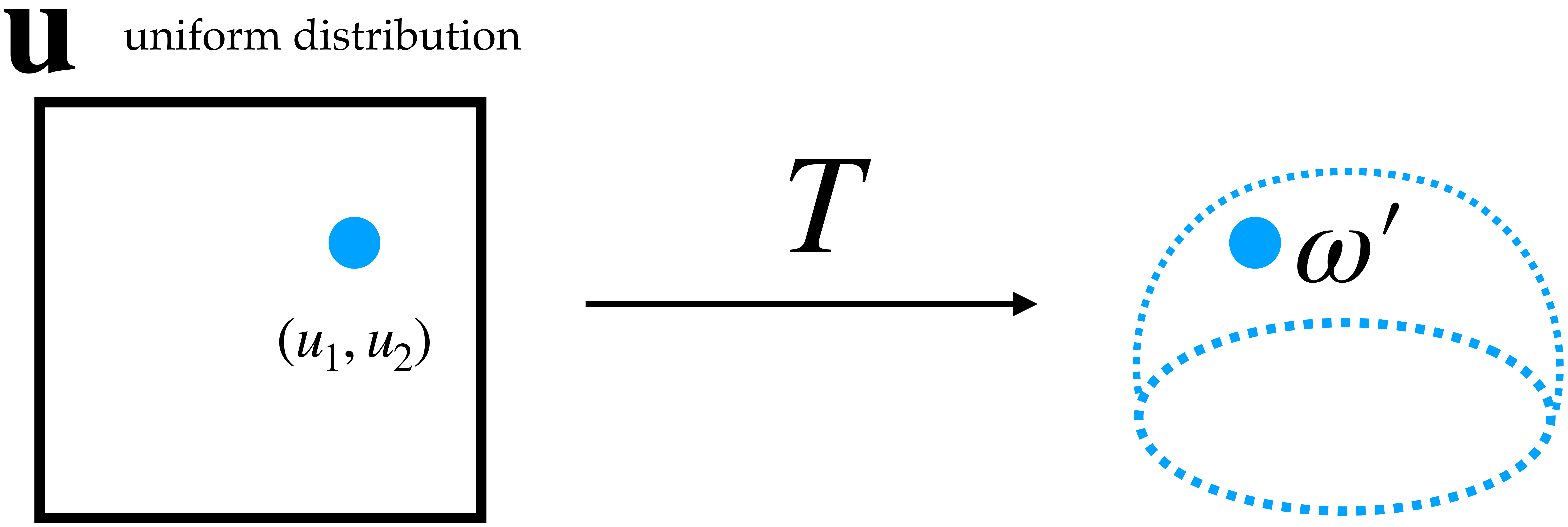
Goal: map **u** to ω'



this mapping is a **change of variable** of our hemispherical integral

$$\iint L'(\omega')\,|\,\omega'\cdot n\,|\,\mathrm{d}\omega' = \iint L'(\omega')\frac{|\,\omega'\cdot n\,|}{\left|\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}\omega'}\right|}\mathrm{d}\mathbf{u}$$

Cosine-weighted hemisphere sampling: choose a mapping with Jacobian $1/\left|n \cdot \omega'\right|$

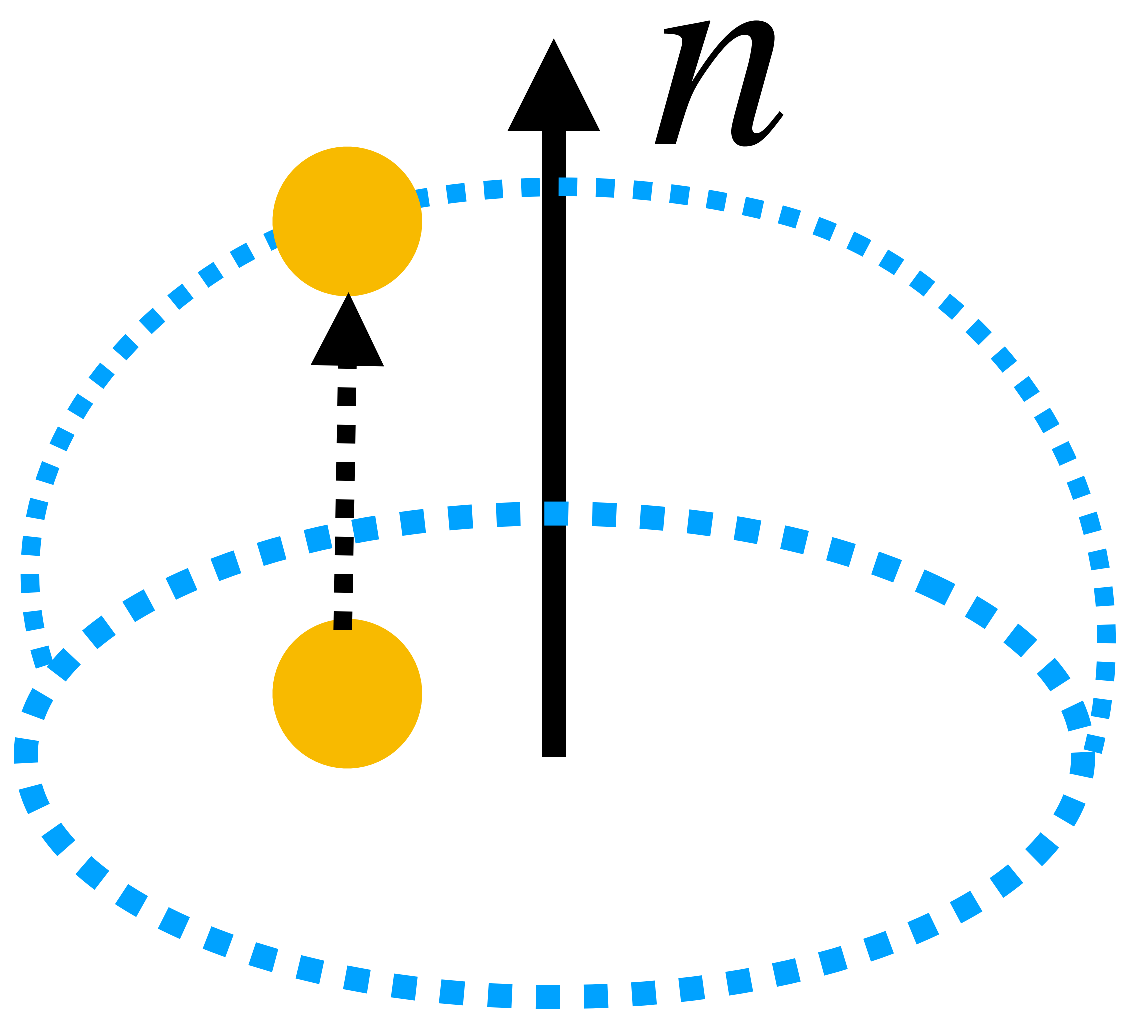


this mapping is a **change of variable** of our hemispherical integral

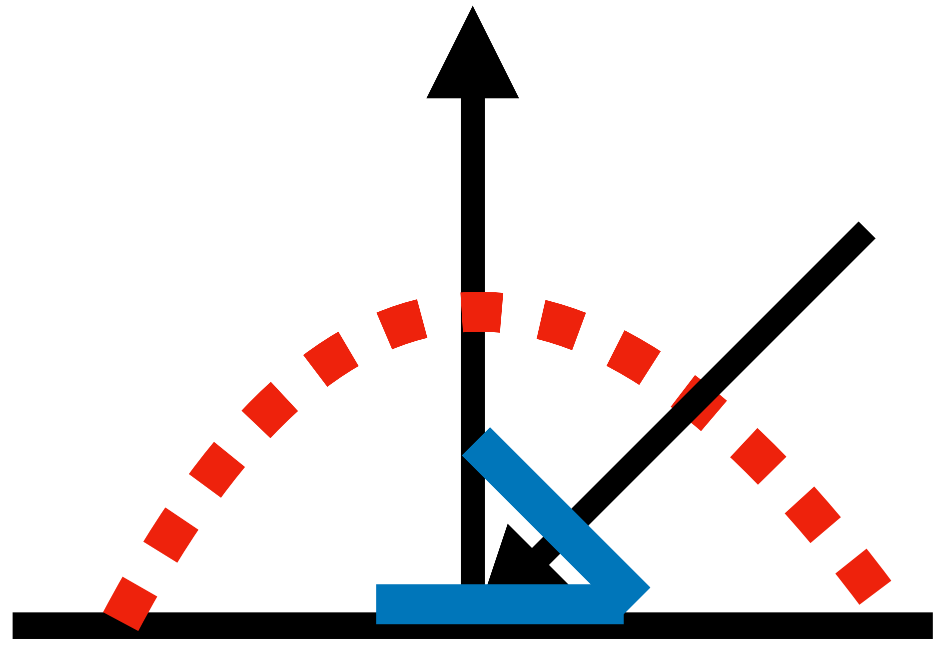
$$\iint L'(\omega')\,|\,\omega' \cdot n\,|\,\mathrm{d}\omega' = \iint L'(\omega')\frac{|\,\omega' \cdot n\,|}{\left|\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}\omega'}\right|}\mathrm{d}\mathbf{u} = \iint L'(\omega')\mathrm{d}\mathbf{u}$$

Malley's method:
sampling cosine-weighted hemisphere by projection

- uniformly sample a point on a disk
- project the point on the hemisphere

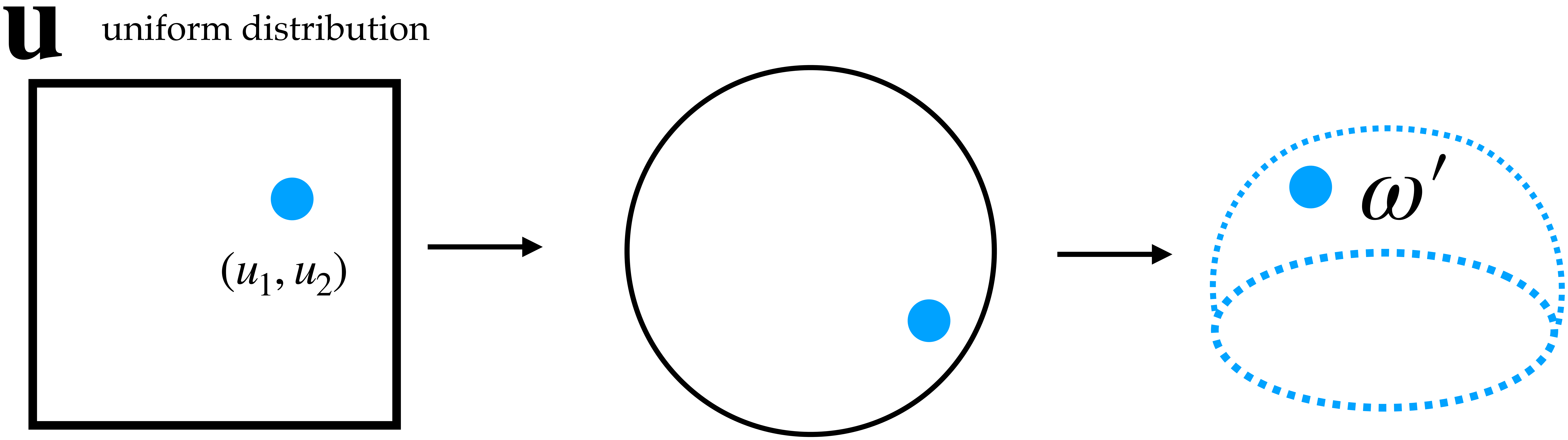


$$\left|\frac{d\omega'}{d\mathbf{u}}\right| = 1/\left|\frac{d\mathbf{u}}{d\omega'}\right| = \pi/\left|\omega' \cdot n\right|$$



geometric intuition:
the cosine term $|\omega' \cdot n|$ is the ratio between an area on hemisphere and an area on surface
so the Jacobian of the mapping is that ratio divided by the area of a unit disk.

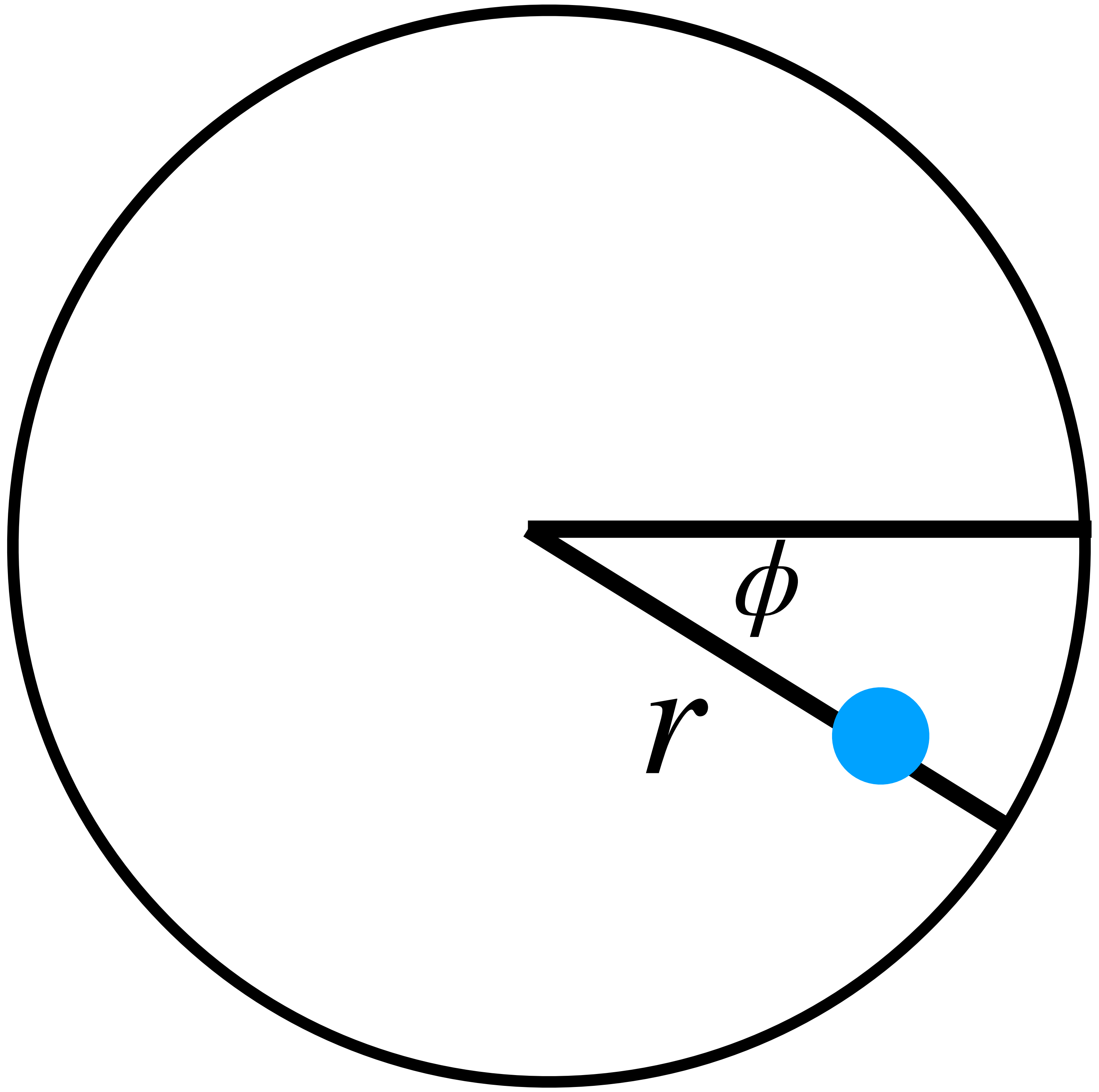
Goal: map $\mathbf{u}$ to ω'



Uniformly sampling a unit disk

- incorrect approach:
  - uniformly pick a distance r from origin
  - uniformly pick an angle ϕ

why is this wrong?



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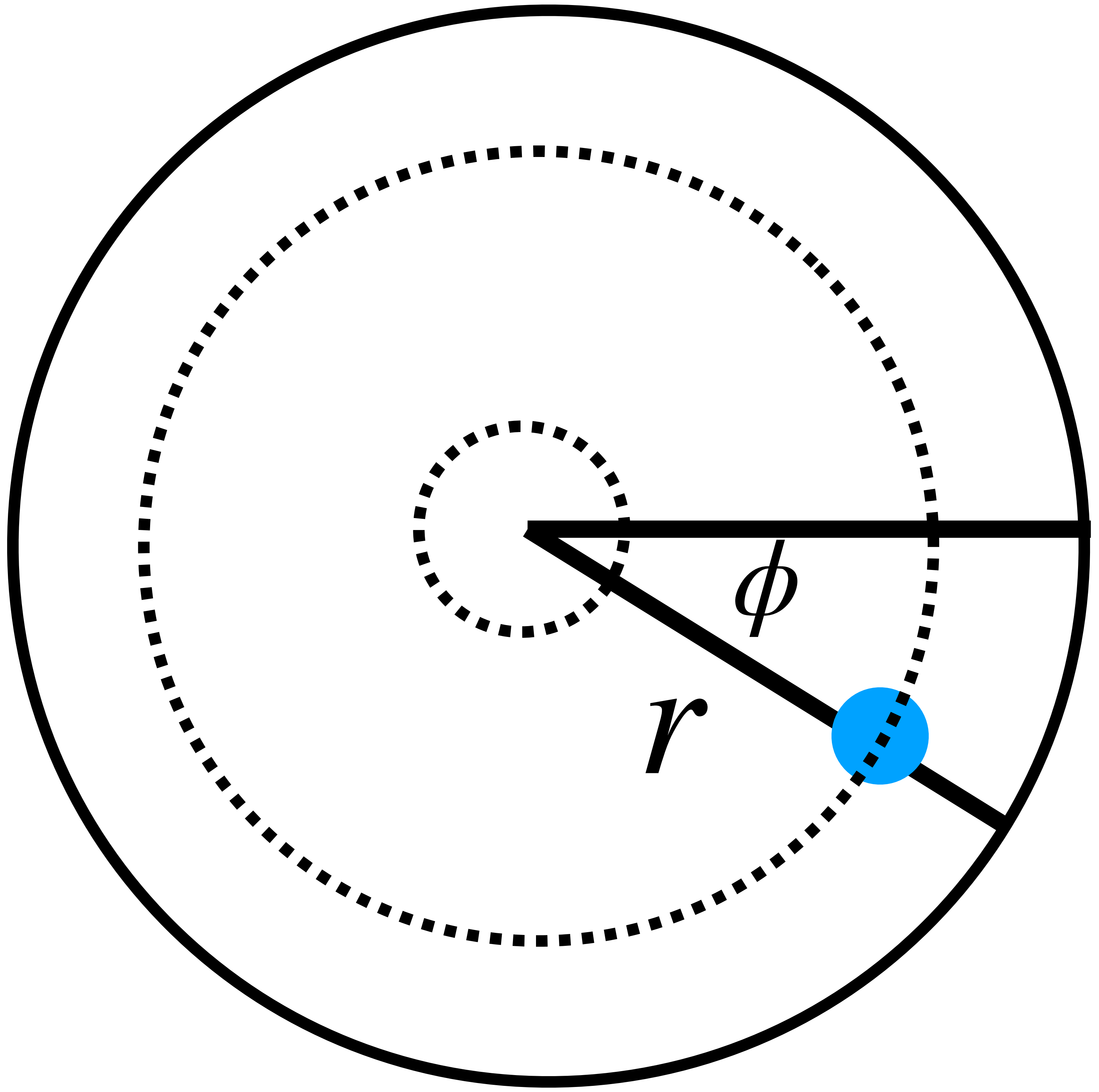


“inner” circles have less area compared to “outer” circles!

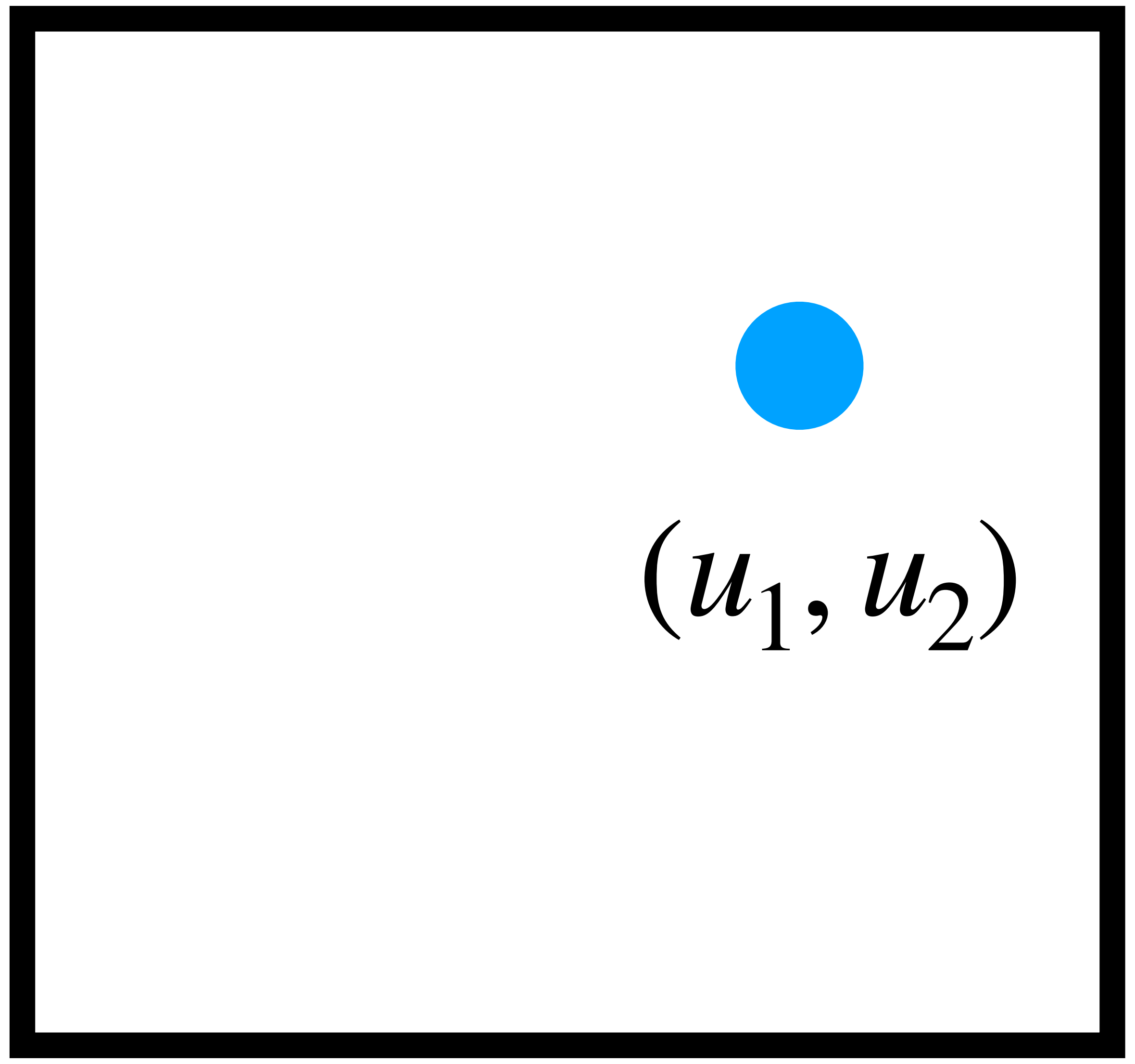
Uniformly sampling a unit disk using change of variable

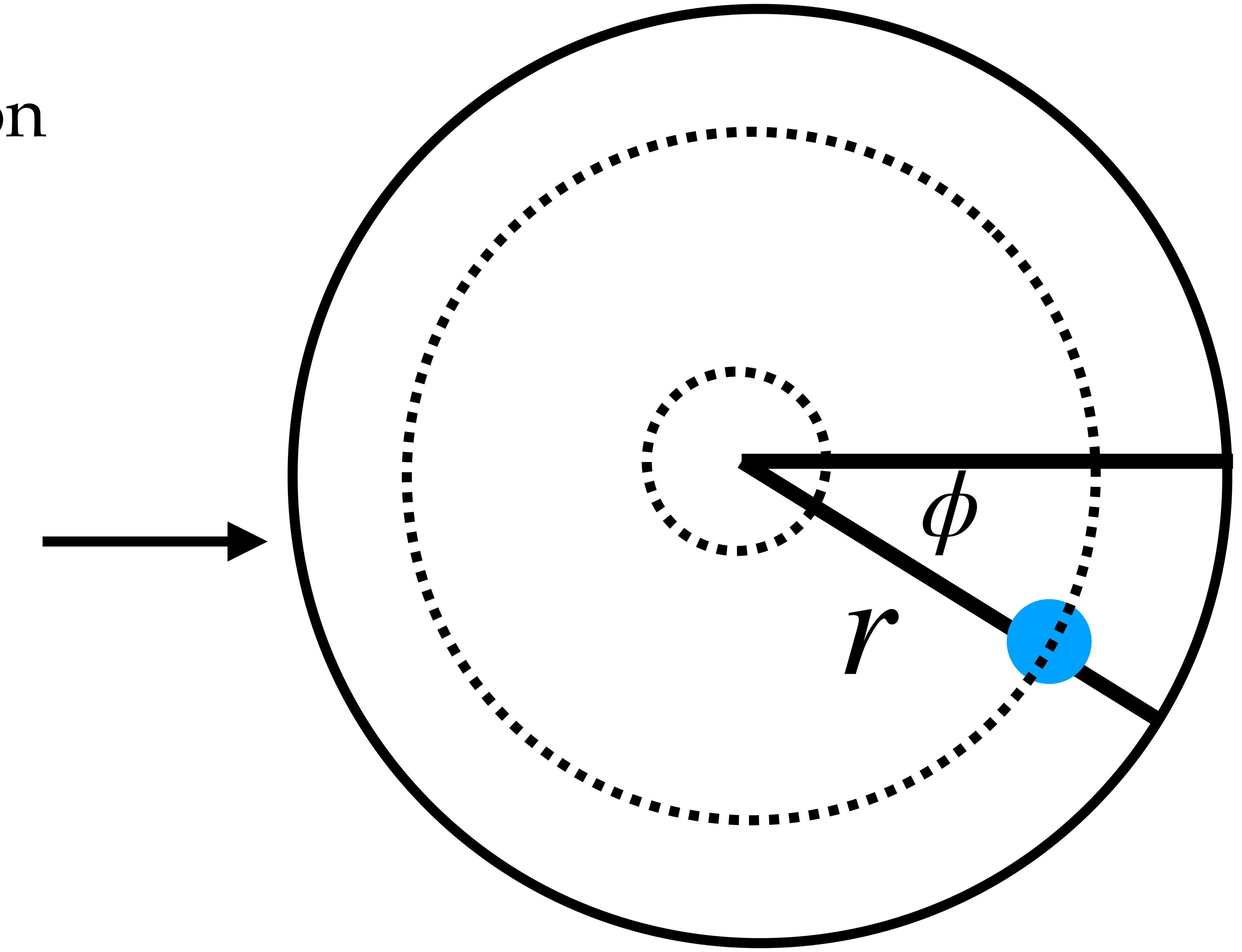
disk area integral

$$\iint r \mathrm{d}r \mathrm{d}\phi$$

$$\iint \mathrm{d}u_1 \mathrm{d}u_2$$

$\mathbf{u}$ uniform distribution





Strategy: one dimension at a time (deal with r first)

disk area integral

Goal: find a transformation $r = T_1(u_1)$ s.t. $\frac{\mathrm{d}u_1}{\mathrm{d}r} \propto r$

$$\iint r\mathrm{d}r\mathrm{d}\phi = 2\pi \int r\mathrm{d}r$$

$$= 2\pi \int \frac{r}{\frac{\mathrm{d}u_1}{\mathrm{d}r}} \mathrm{d}u_1$$

Strategy: one dimension at a time (deal with r first)

disk area integral

Goal: find a transformation $u_1 = T_1^{-1}(r)$ s.t. $\frac{\mathrm{d}u_1}{\mathrm{d}r} \propto r$

$$\iint r \mathrm{d}r \mathrm{d}\phi = 2\pi \int r \mathrm{d}r$$

$$= 2\pi \int \frac{r}{\frac{\mathrm{d}u_1}{\mathrm{d}r}} \mathrm{d}u_1$$

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$$\iint r\mathrm{d}r\mathrm{d}\phi = 2\pi \int r\mathrm{d}r$$

$$u_1 \propto \int_0^r r'\mathrm{d}r' = \frac{1}{2}r^2 \qquad u_1 = \frac{A}{2}r^2$$

$$= 2\pi \int \frac{r}{\frac{\mathrm{d}u_1}{\mathrm{d}r}}\mathrm{d}u_1$$

Strategy: one dimension at a time (deal with r first)

disk area integral

$$\iint r\mathrm{d}r\mathrm{d}\phi = 2\pi\int r\mathrm{d}r$$

$$= 2\pi\int \frac{r}{\frac{\mathrm{d}u_1}{\mathrm{d}r}}\mathrm{d}u_1$$

Goal: find a transformation $u_1 = T_1^{-1}(r)$ s.t. $\frac{\mathrm{d}u_1}{\mathrm{d}r} \propto r$

$$u_1 \propto \int_0^r r'\mathrm{d}r' = \frac{1}{2}r^2 \qquad u_1 = \frac{A}{2}r^2$$

add constraints $T_1^{-1}(0) = 0, T_1^{-1}(1) = 1$
(so that $u_1 \in [0,1]$)

$$A = 2$$

Strategy: one dimension at a time (deal with r first)

disk area integral

Goal: find a transformation $u_1 = T_1^{-1}(r)$ s.t. $\frac{\mathrm{d}u_1}{\mathrm{d}r} \propto r$

$$\iint r \mathrm{d}r \mathrm{d}\phi = 2\pi \int r \mathrm{d}r$$

$$u_1 = r^2$$

$$= 2\pi \int \frac{r}{\frac{\mathrm{d}u_1}{\mathrm{d}r}} \mathrm{d}u_1$$

Strategy: one dimension at a time (deal with r first)

disk area integral

Goal: find a transformation $u_1 = T_1^{-1}(r)$ s.t. $\frac{\mathrm{d}u_1}{\mathrm{d}r} \propto r$

$$\iint r \mathrm{d}r \mathrm{d}\phi = 2\pi \int r \mathrm{d}r$$

$$= 2\pi \int \frac{r}{\frac{\mathrm{d}u_1}{\mathrm{d}r}} \mathrm{d}u_1$$

$$u_1 = r^2$$

$$r = \sqrt{u_1}$$

Strategy: one dimension at a time (deal with ϕ next)

disk area integral

Goal: find a transformation $u_1 = T_1^{-1}(r)$ s.t. $\frac{\mathrm{d}u_1}{\mathrm{d}r} \propto r$

$$\iint r \mathrm{d}r \mathrm{d}\phi = 2\pi \int r \mathrm{d}r$$

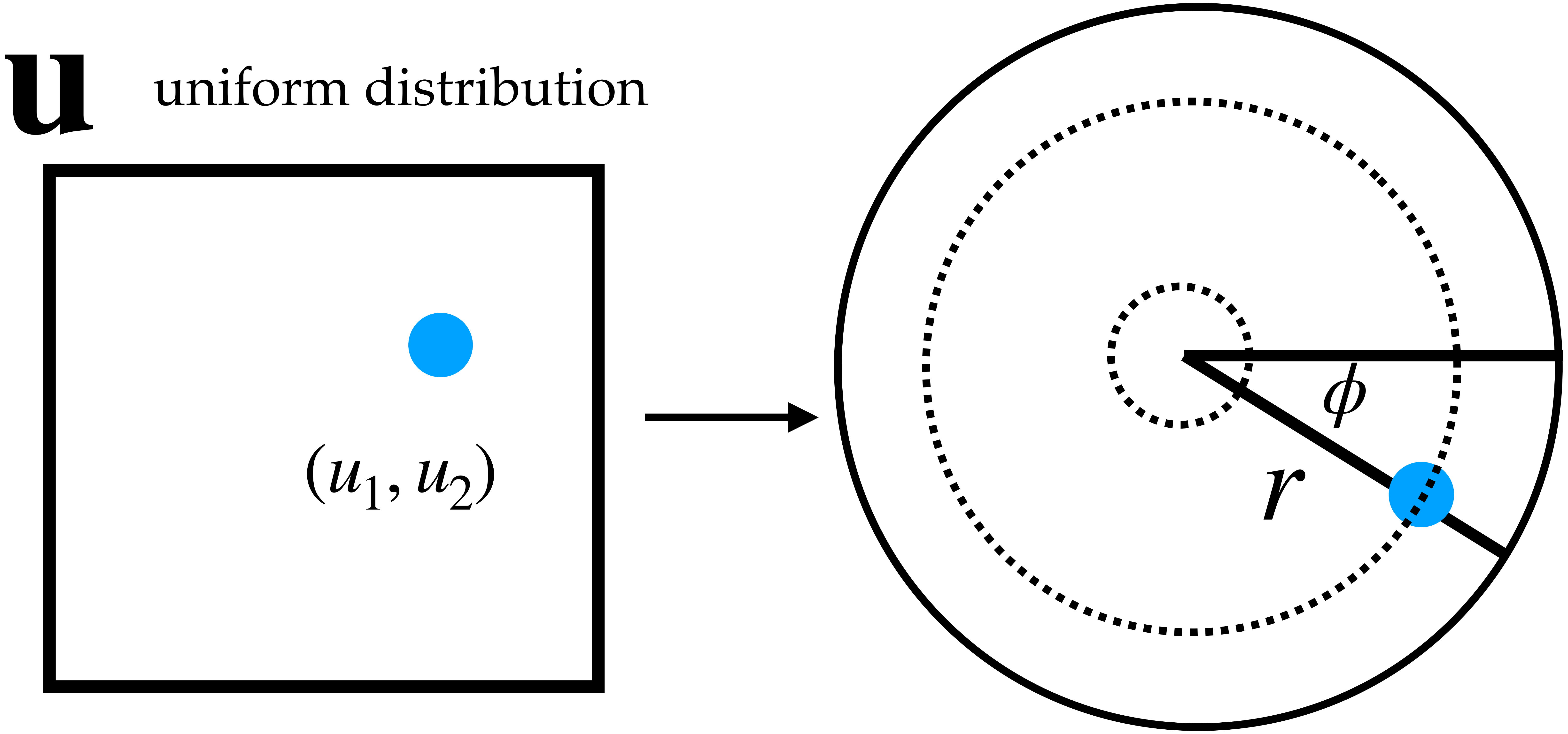
$$= 2\pi \int \frac{r}{\frac{\mathrm{d}u_1}{\mathrm{d}r}} \mathrm{d}u_1$$

$$u_1 = r^2$$

$$r = \sqrt{u_1}$$

$$\phi = 2\pi u_2$$

Mapping a square to a uniform disk



$$r = \sqrt{u_1}$$

$$\phi = 2\pi u_2$$

Malley's method:
sampling cosine-weighted hemisphere by projection

- uniformly sample a point on a disk
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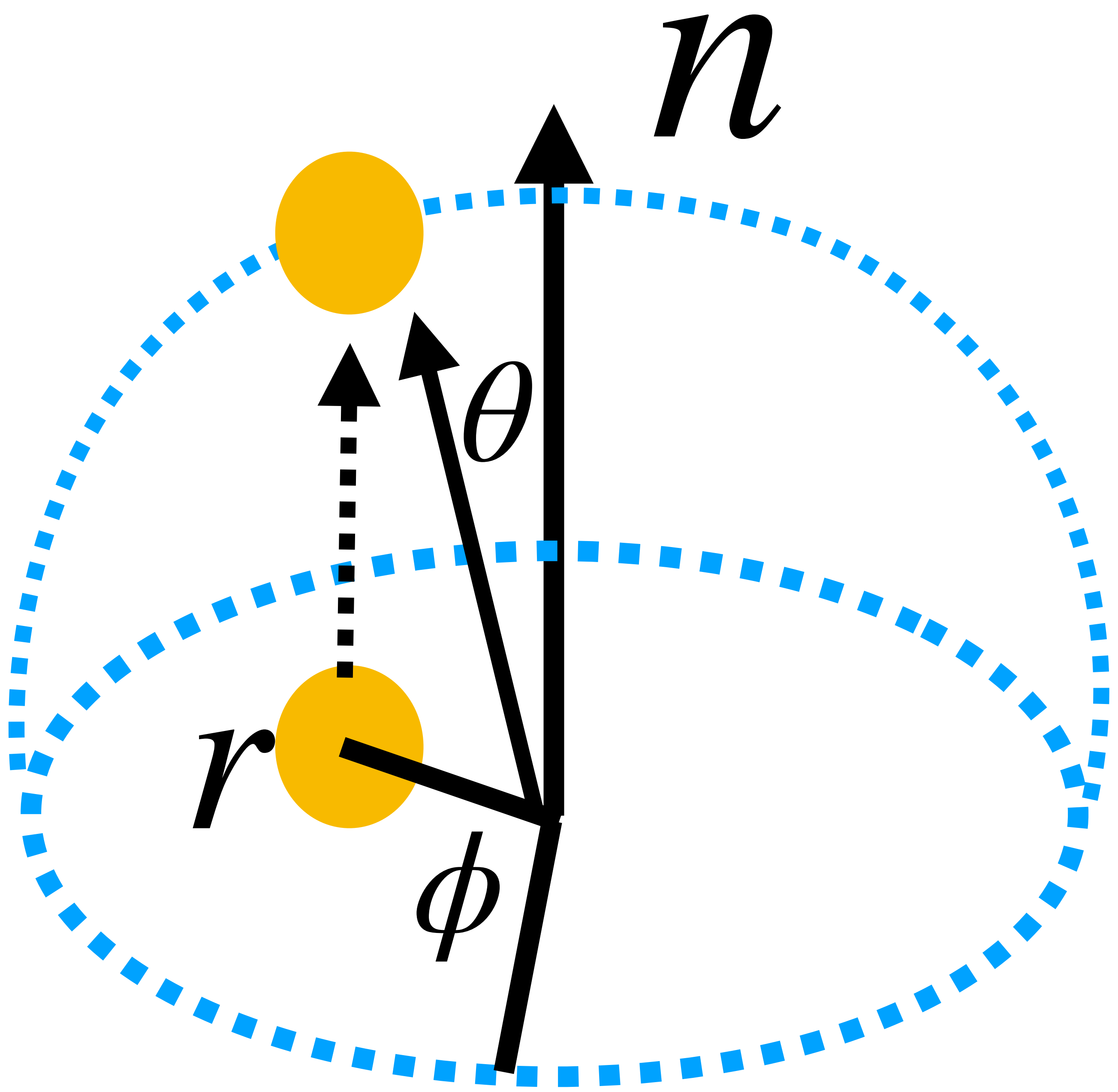

$$r = \sqrt{u_1}$$
$$\phi = 2\pi u_2$$

$$\cos\theta = \sqrt{1 - r^2}$$

exercise: compute the Jacobian and show the correctness

Cosine-weighted hemisphere sampling allows us to approximate integrals

$$\iint L'(\omega')\,|\,\omega' \cdot n\,|\,\mathrm{d}\omega'$$

$$\approx \frac{1}{N}\sum_{i=1}^{N} \frac{L'(\omega'(\mathbf{u}_i))\,|\,\omega'(\mathbf{u}_i)\cdot n\,|}{p(\omega'(\mathbf{u}_i))}$$

$$p(\omega') = \frac{\left|\omega' \cdot n\right|}{\pi}$$

probability density function = Jacobian