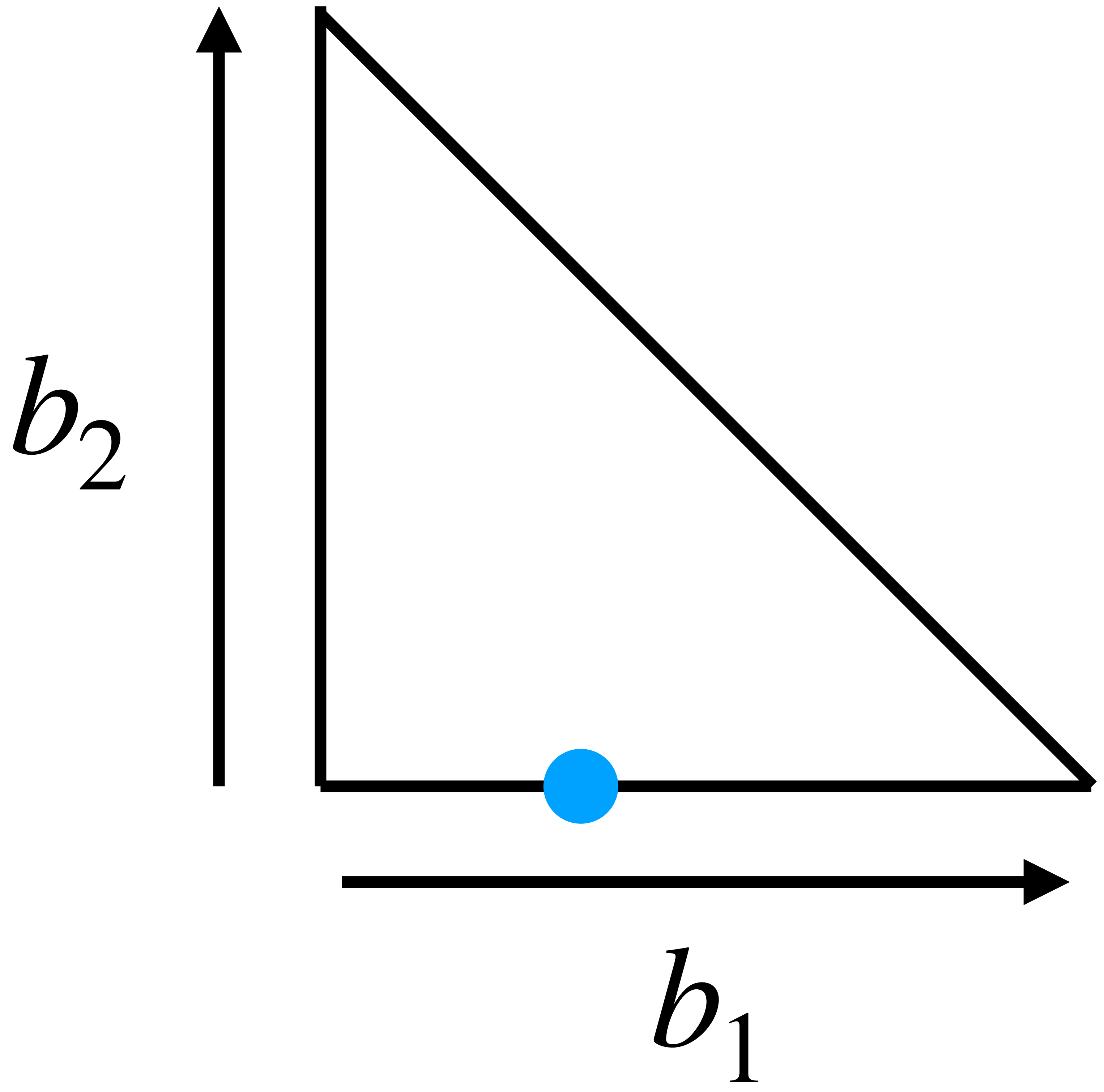
Uniformly sample a point on a triangle



incorrect strategy:

1. uniform sample b1

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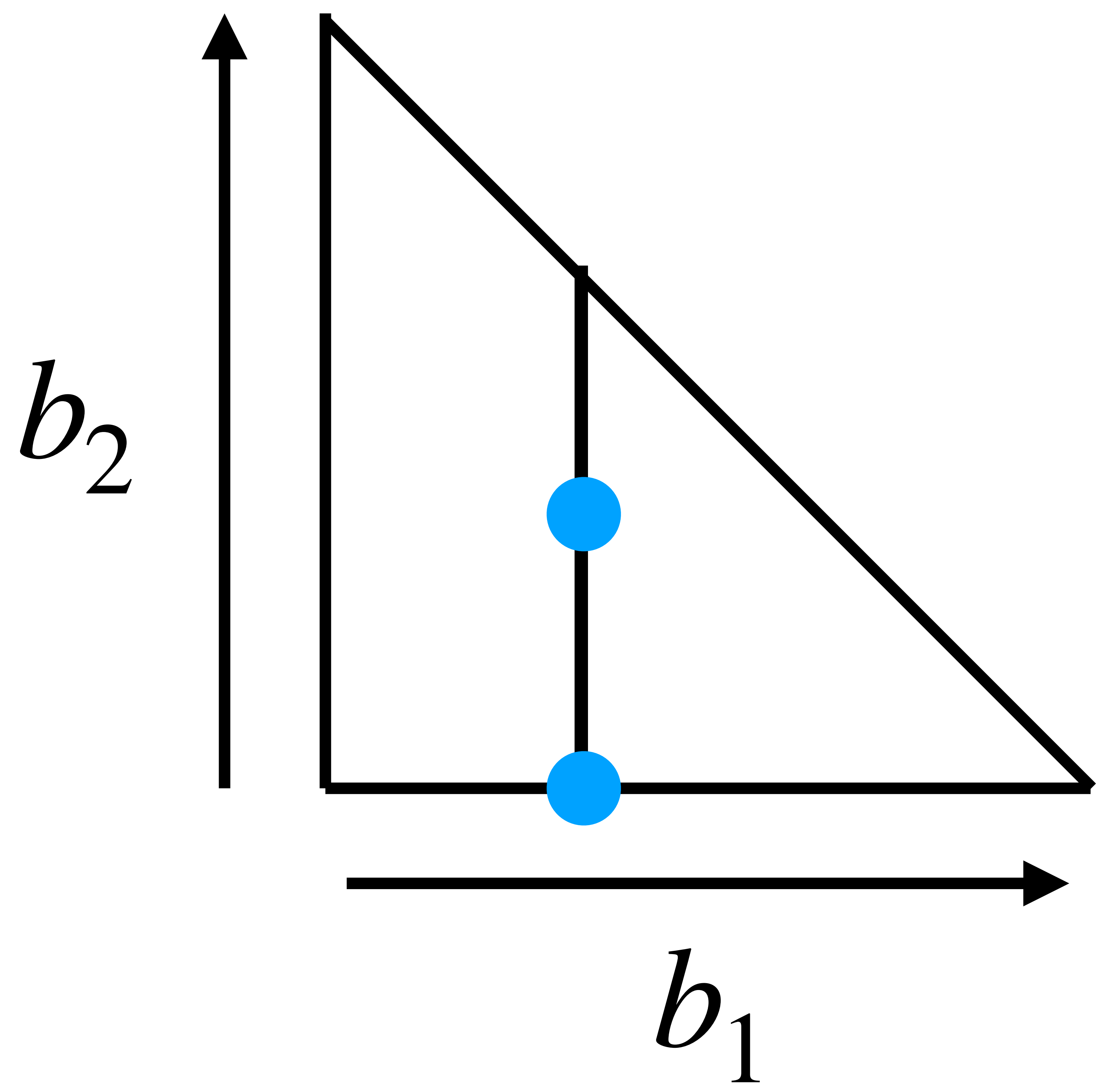
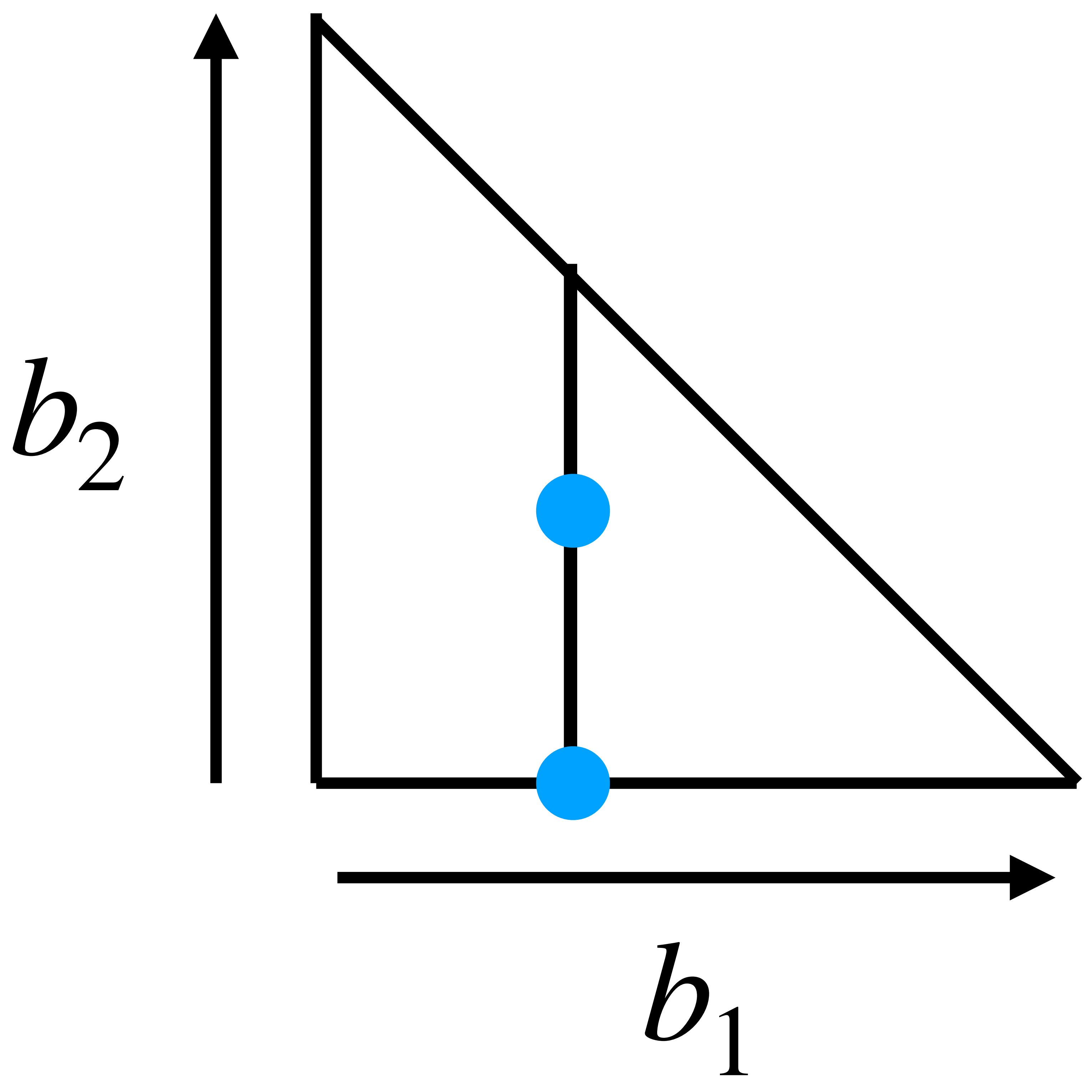


incorrect strategy:

1. uniform sample b1
2. uniform sample b2 ~ U(0, 1 - b1)

Uniformly sample a point on a triangle



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why is this wrong?

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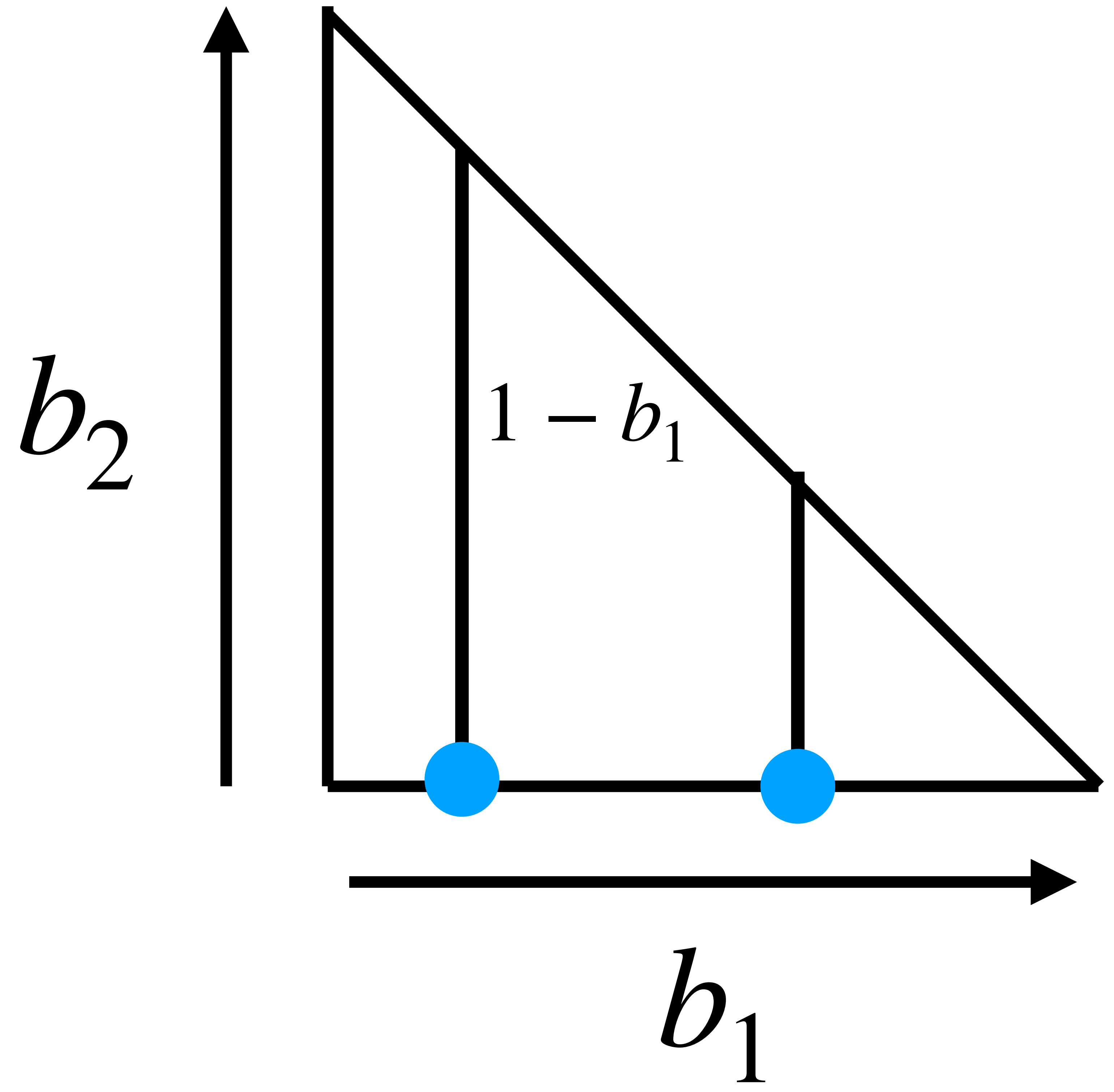


smaller b1 has more area!

Recall: sampling = change of variable

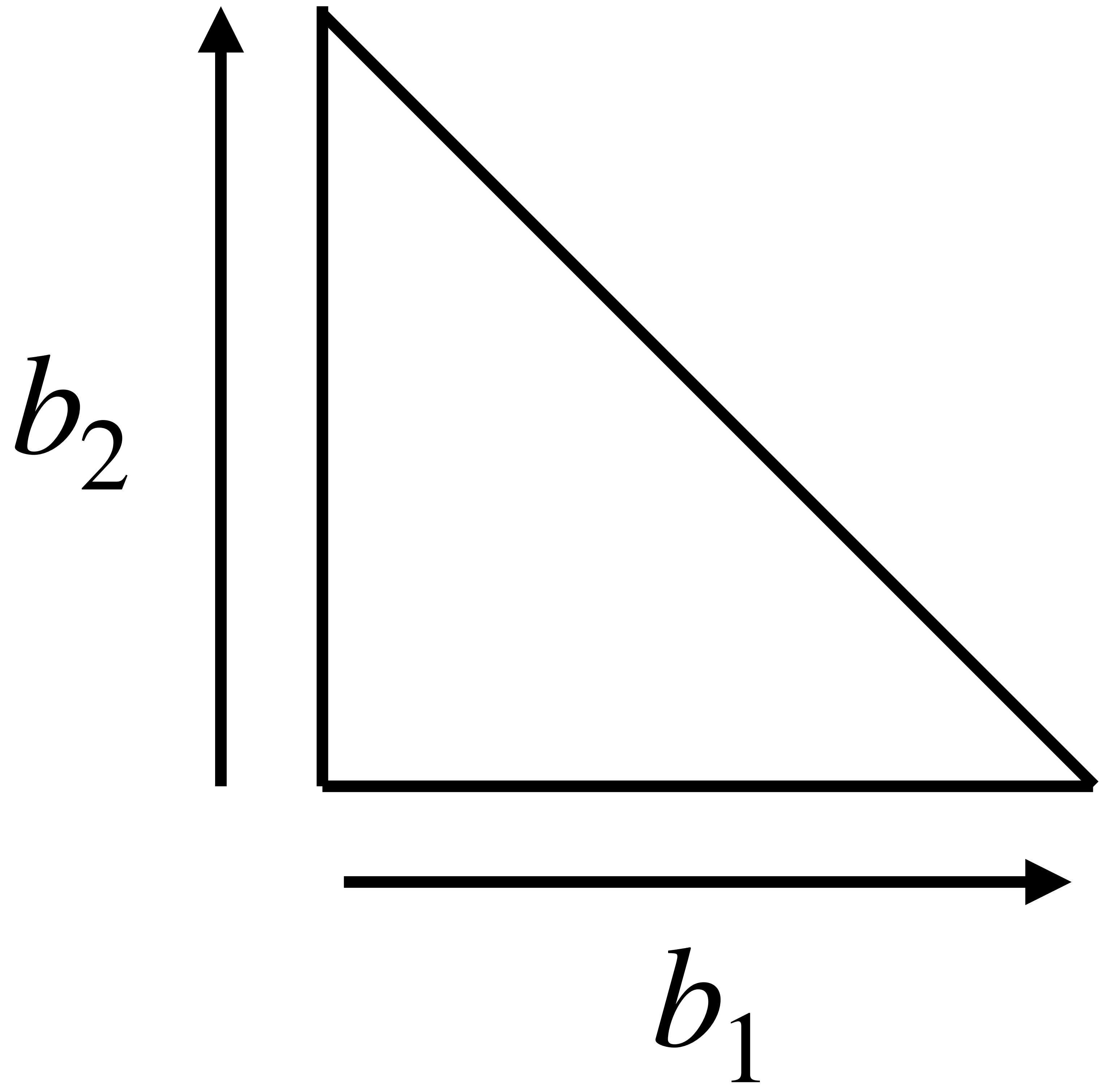
$$\iint A(\mathbf{x})\mathrm{d}\mathbf{x}$$

$$\mathbf{x} = T(\mathbf{u})$$

$$\mathbf{u} = T^{-1}(\mathbf{x})$$

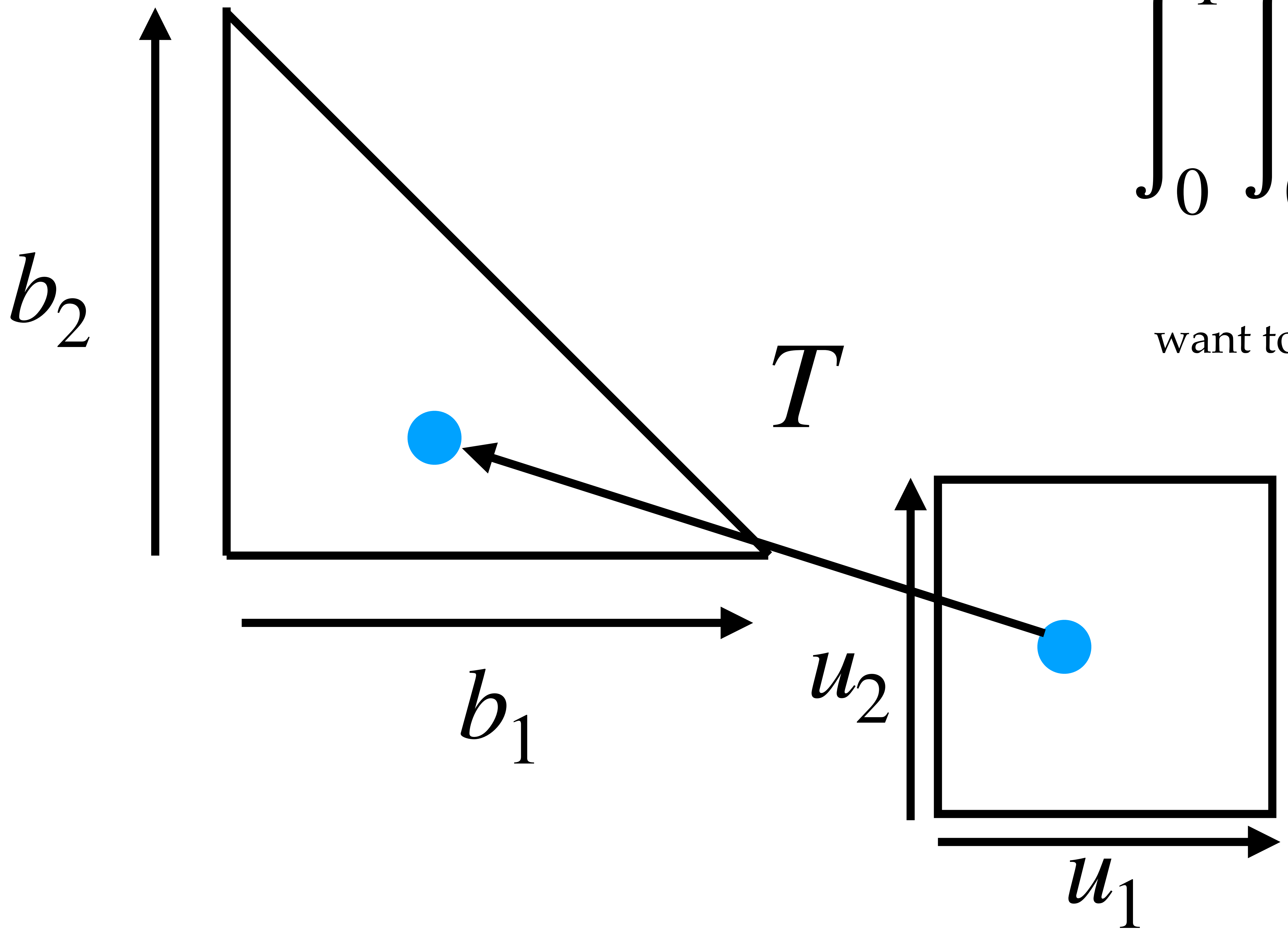
$$= \iint A(\mathbf{x}(\mathbf{u}))\frac{1}{\left|\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}\mathbf{x}}\right|}\mathrm{d}\mathbf{u}$$

A triangular area integral has variable dependent bounds



$$\int_0^1 \int_0^{1-b_1} \mathrm{d}b_2 \mathrm{d}b_1$$

Sampling = mapping a uniform domain to the target



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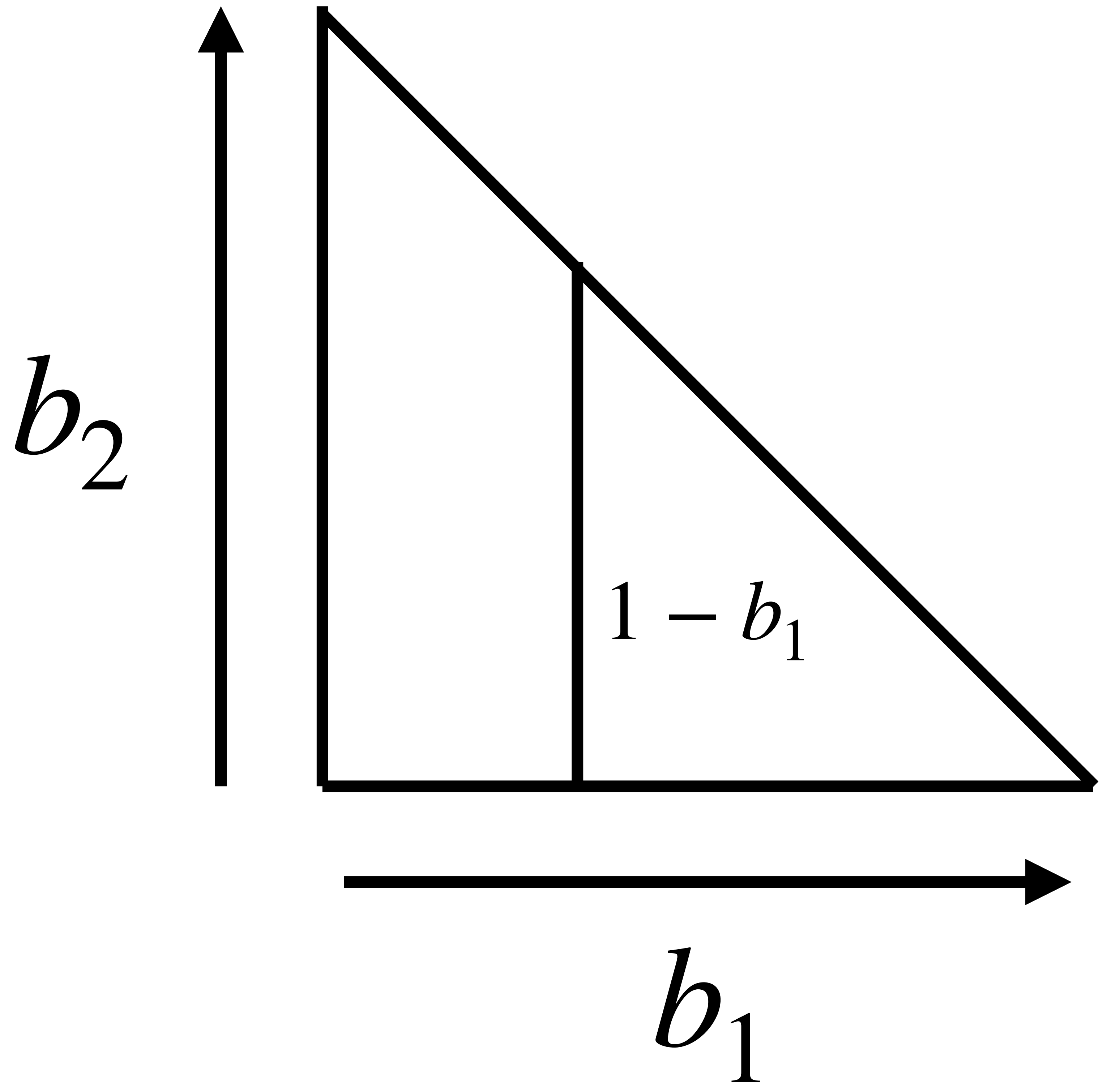
want to map $(u_1, u_2) \in [0,1]^2$ to (b_1, b_2)

$$\int_0^1 \int_0^1 \frac{1}{\left|\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}\mathbf{b}}\right|} \mathrm{d}u_1 \mathrm{d}u_2$$

Strategy: deal with one dimension at a time



we want to sample b1 s.t. $p(b_1) = \left| \frac{\mathrm{d}u_1}{\mathrm{d}b_1} \right| \propto 1 - b_1$

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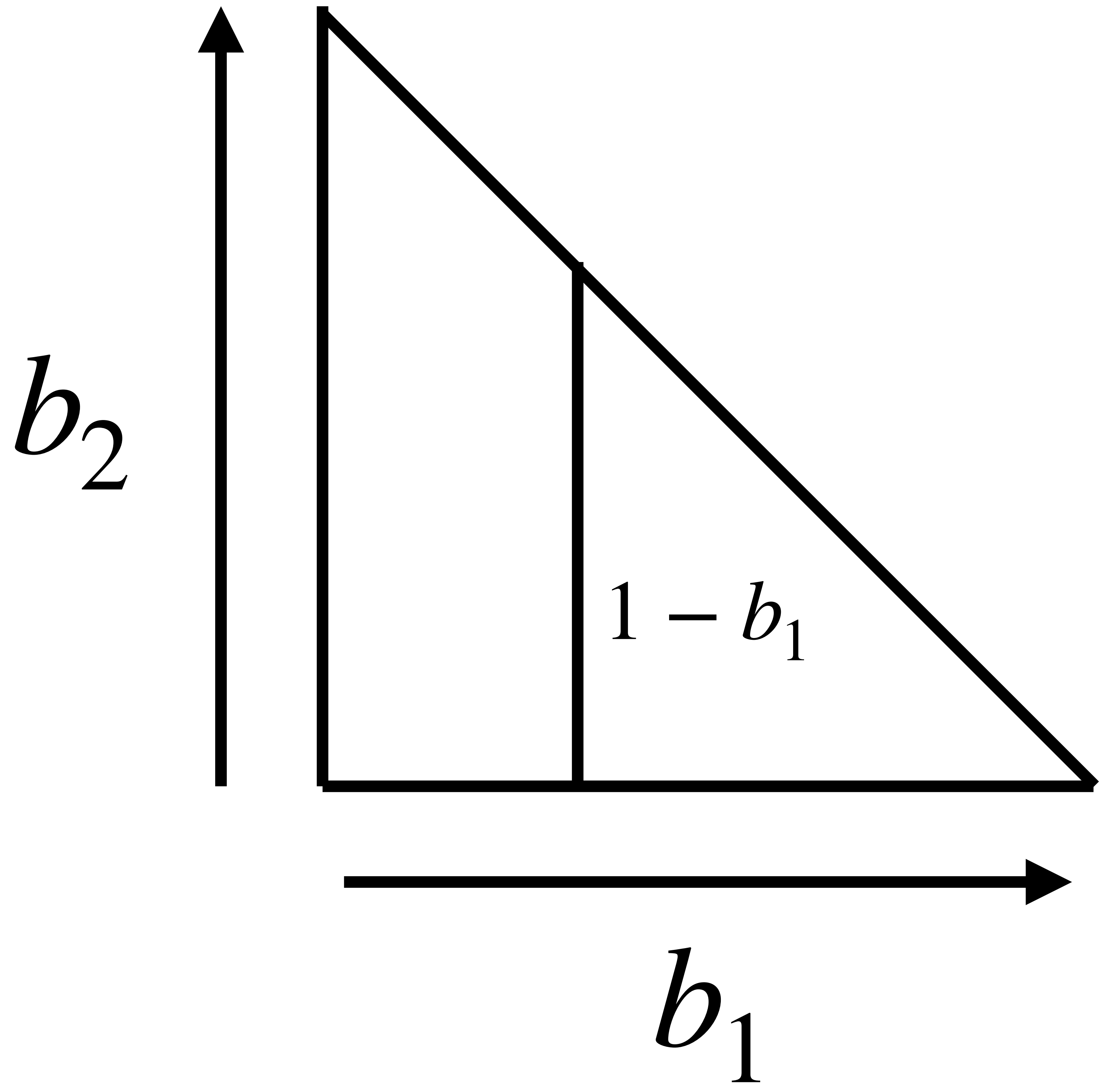

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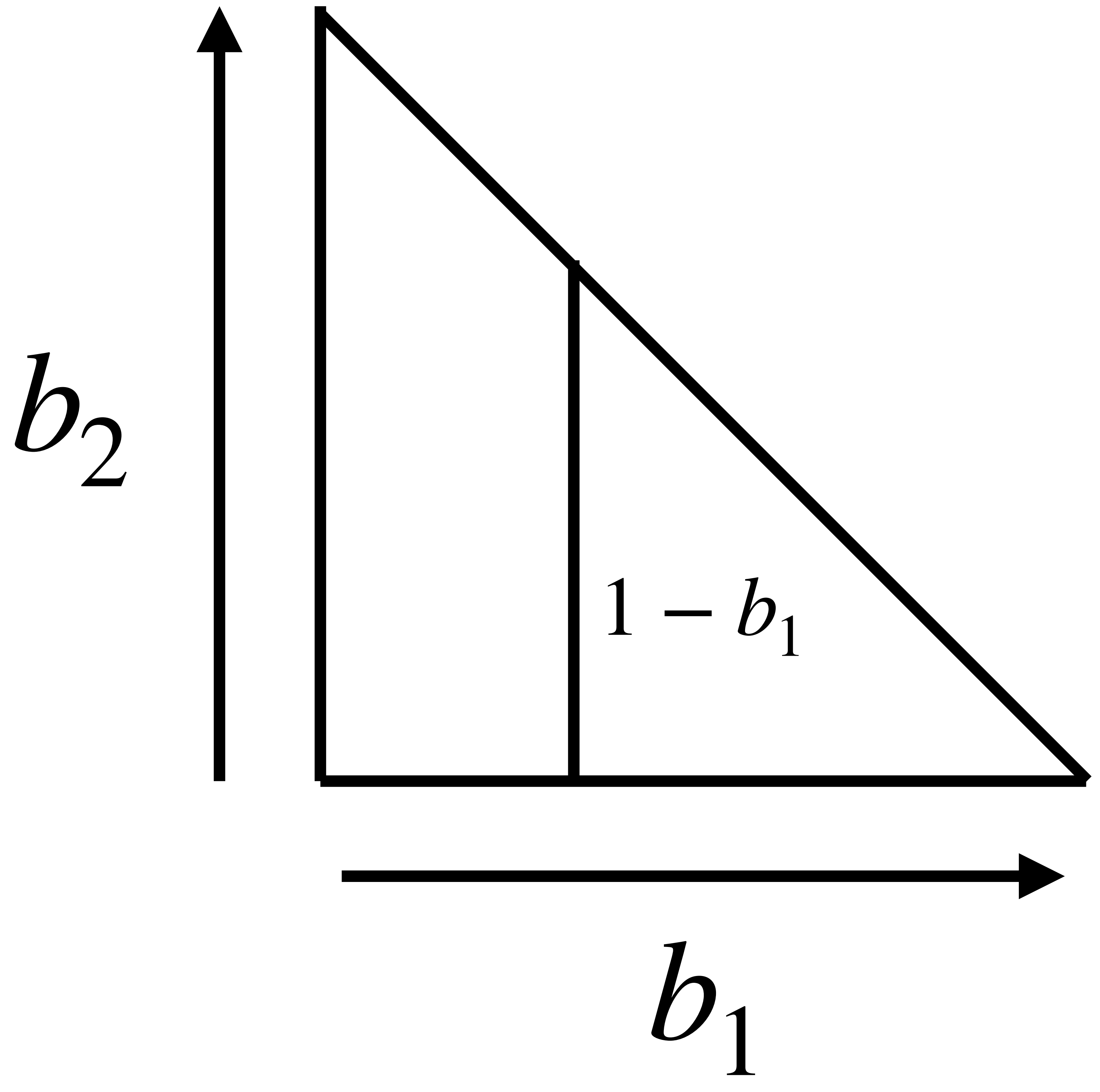

we want to sample b1 s.t. $p(b_1) = \left|\frac{\mathrm{d}u_1}{\mathrm{d}b_1}\right| \propto \left(1 - b_1\right)$

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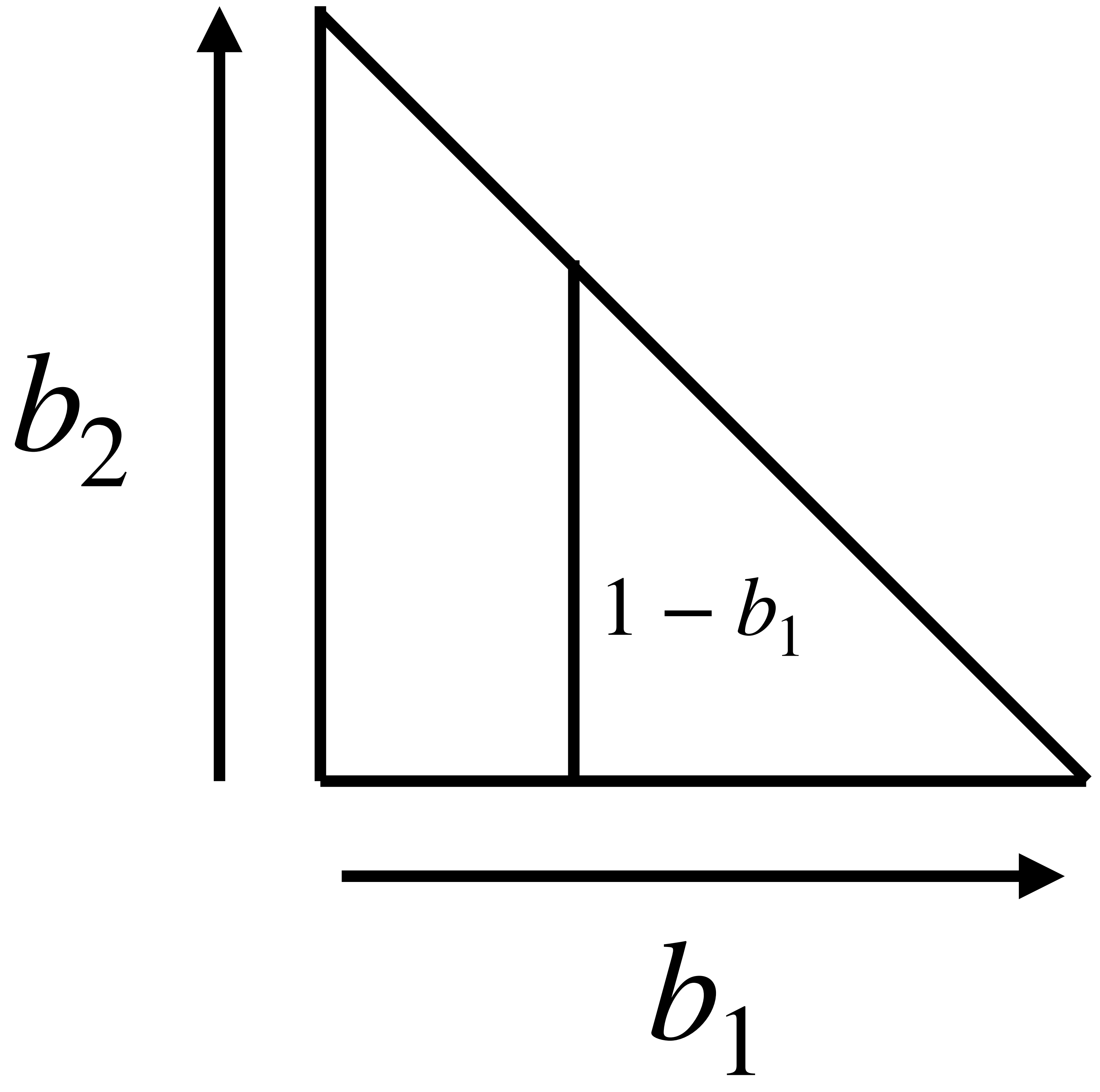

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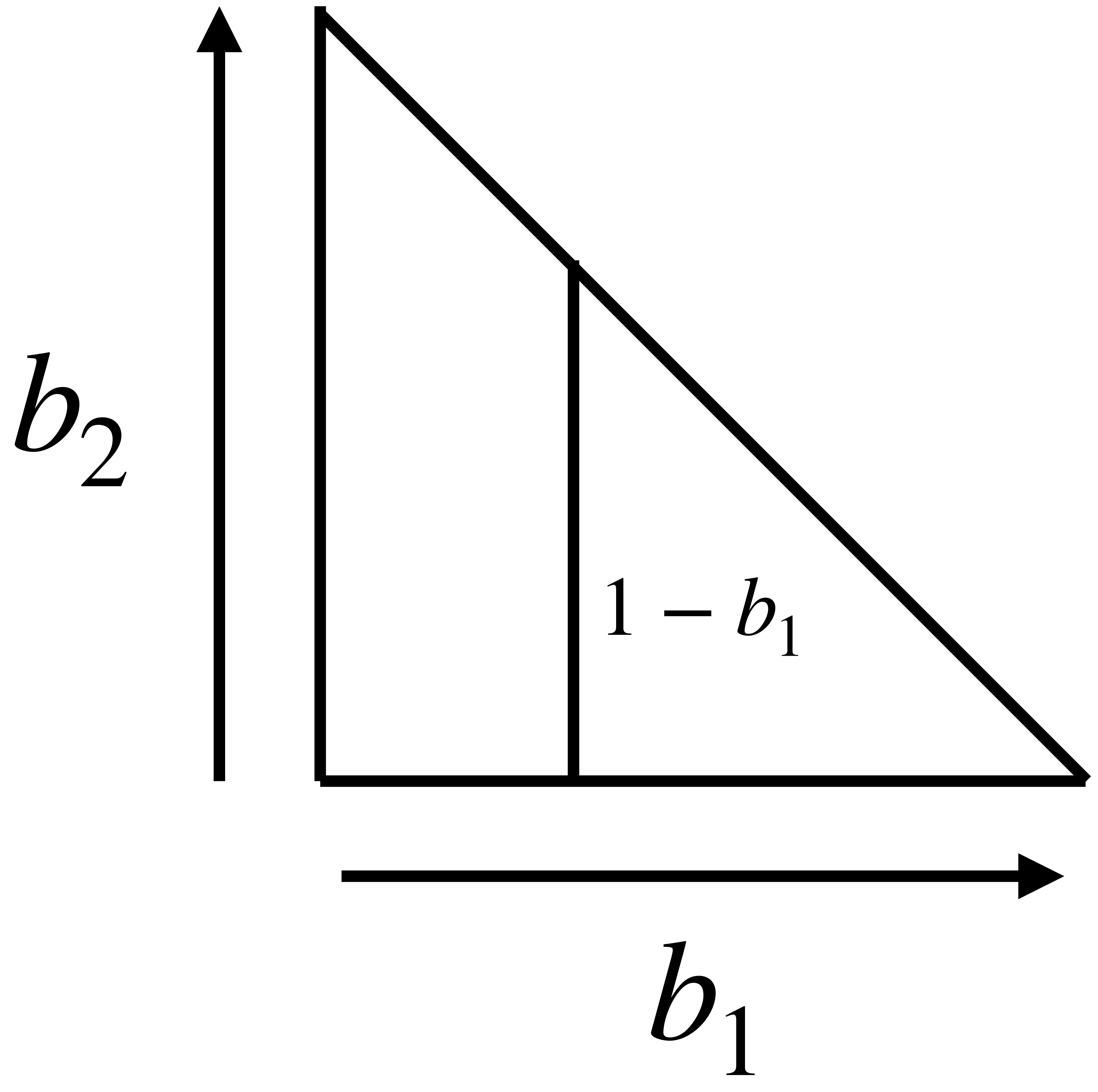

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the anti-derivative!

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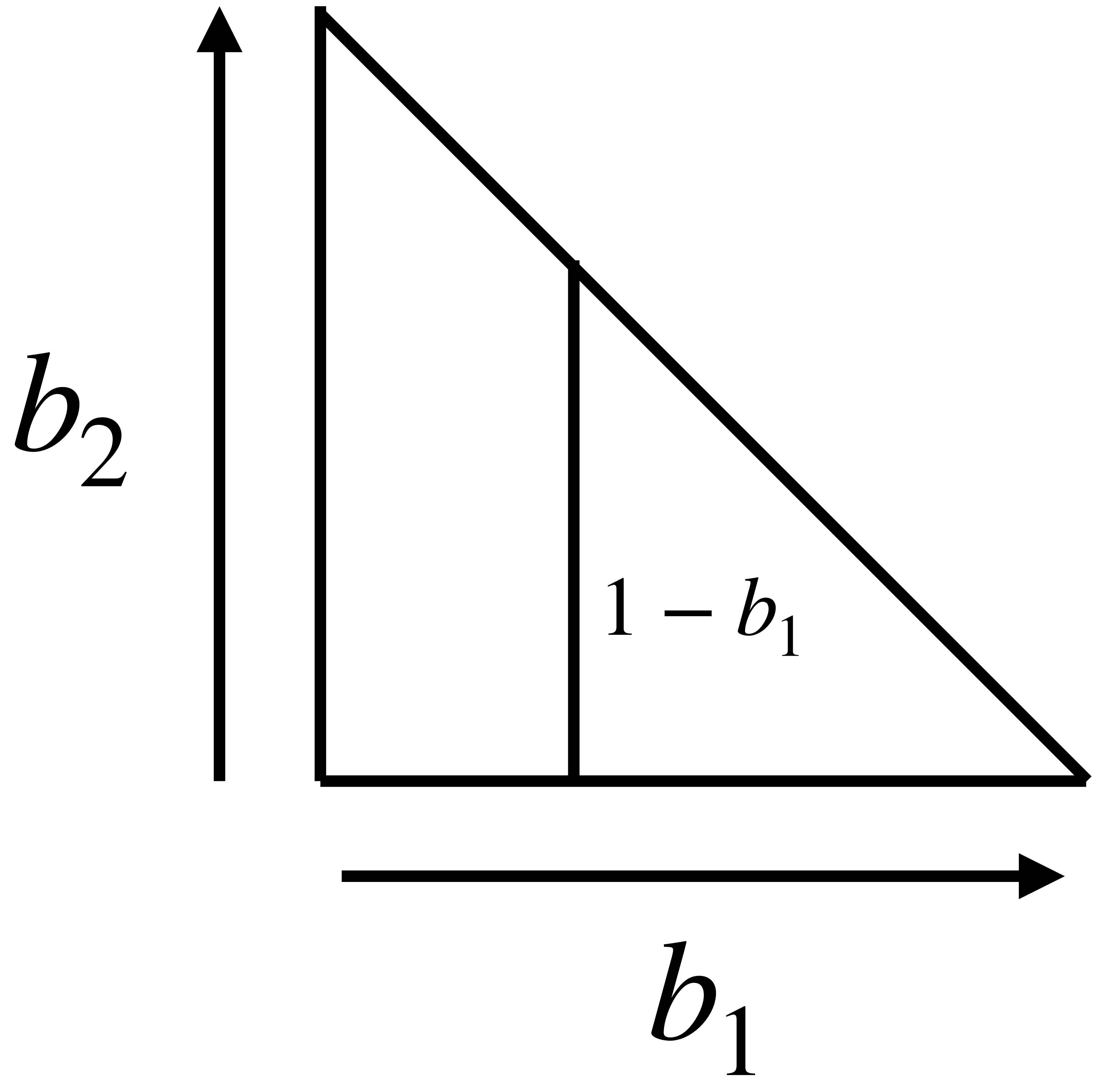

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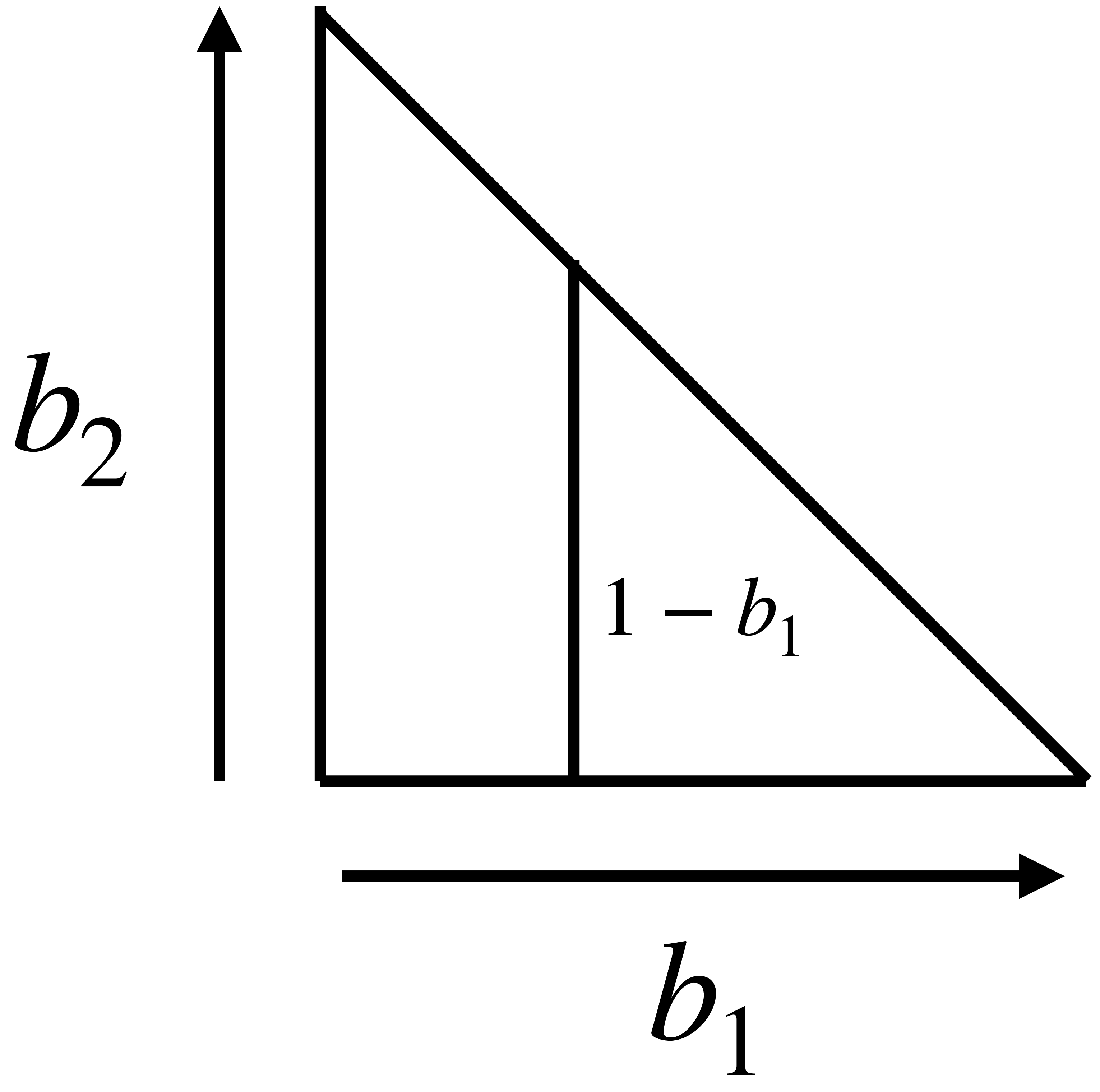

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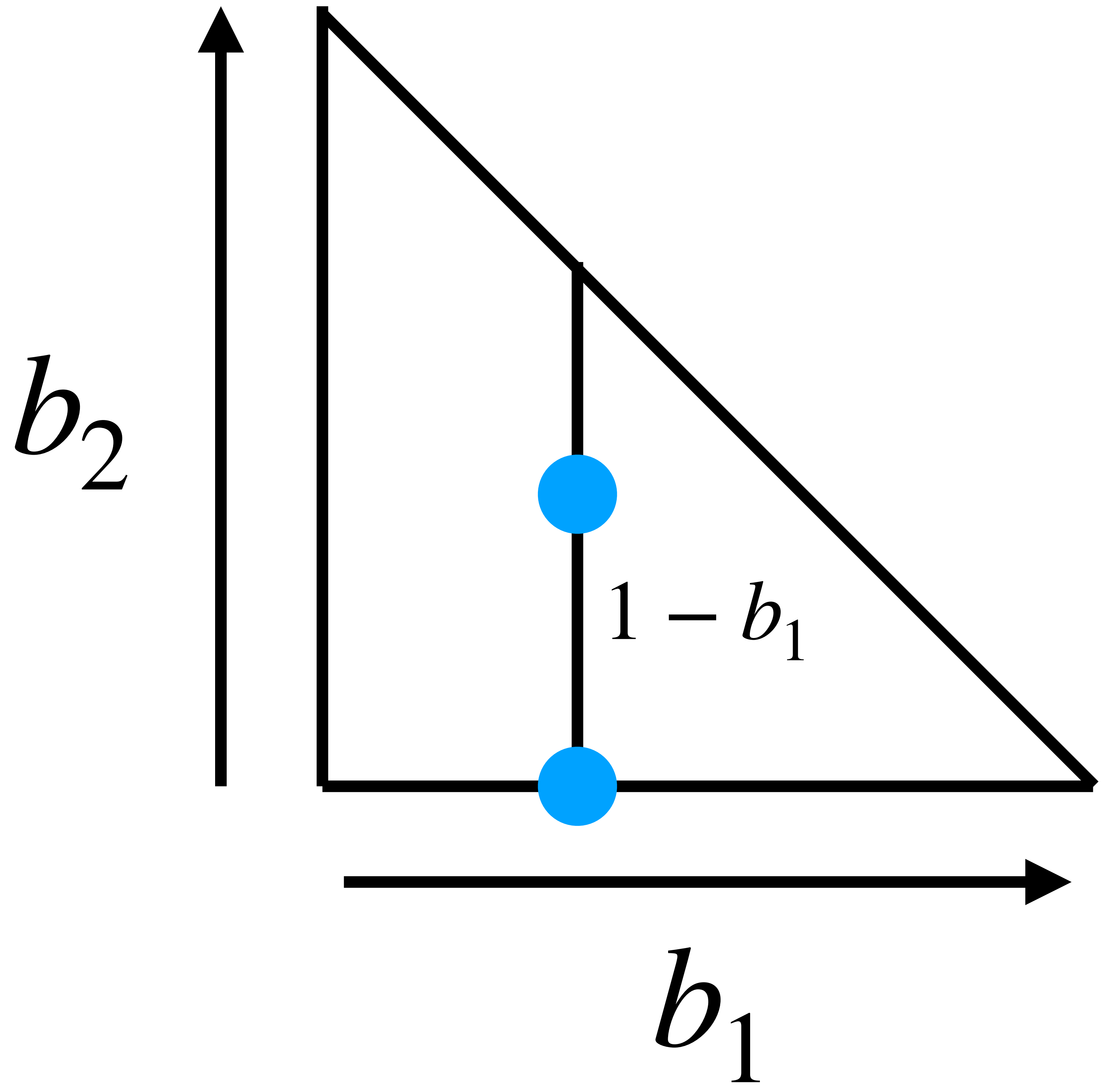
$$b_1 = 1 - \sqrt{u_1}$$

We can uniformly sample b2 after b1 is sampled



$$b_1 = 1 - \sqrt{u_1}$$

$$b_2 = (1 - b_1)u_2$$