Wave Optics with Rays

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Some Applications

Visual Spectrum
• Wave optics BRDFs
• Micro-scale rendering
• Optical instrument design & optimization
• Rendering of polarized light, materials, filters, birefringence, etc.

Non-Visual
• Sound propagation
• Electromagnetic propagation (radar, radio communications, etc.)
Wave Optics

- Color spectrum
- Dispersion
- Polarization
- Interference
- Diffraction
- Coherence

Maxwell’s Equations (macroscopic SI form)

\[
\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \\
\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \\
\n\nabla \cdot \mathbf{D} = \rho \\
\n\nabla \cdot \mathbf{B} = 0 \\
\n\]

- \( \mathbf{E} \) electric field
- \( \mathbf{H} \) magnetizing field
- \( \mathbf{J} \) electric current density
- \( \rho \) electric charge density
- \( \mu \) material permeability
- \( \varepsilon \) material permittivity
Computational Electromagnetics

• Finite difference time domain (FDTD)
• Method of moments (MoM)
• Boundary element method (BEM)
• Finite element method (FEM)
• Fast multipole method (FMM)
• Geometric optics (GO)
• Physical optics (PO)
• Uniform theory of diffraction (UTD)
Computational Electromagnetics

- Finite difference time domain (FDTD)  
  Grid based
- Method of moments (MoM)  
  Element based
- Boundary element method (BEM)  
  Element based
- Finite element method (FEM)  
  Element based
- Fast multipole method (FMM)  
  Element/particle based
- Geometric optics (GO)  
  Ray based
- Physical optics (PO)  
  Element/ray based
- Uniform theory of diffraction (UTD)  
  Ray based
Geometric Optics & Wavefront Tracing
Huygens-Fresnel Principle

• A wavefront can be represented as an infinite number of interfering point sources
Eikonal Equation

\[ |\nabla \Psi(x)| = n(x) \]

- Phase function: \( \Psi(x) \)
- Index of refraction: \( n(x) \)

- Wavefront propagates along gradient of phase function (i.e., along the normal of the wavefront)
Geometric Optics

- Geometric optics models the propagation of light (or EM fields) in terms of rays
- Main assumptions include:
  - Light travels along straight-line paths in a homogeneous medium
  - Fresnel type surfaces (i.e., smooth metals and dielectrics – and no rough or diffuse surfaces)
  - Wavelength of light is much smaller than smallest geometric feature

- The key equations of GO can be derived directly from Maxwell’s equations, by applying a variety of approximations based on these assumptions
Relative Frequency

- In *low frequency* situations (i.e., when the wavelength is on the same scale or larger than the geometric features), the diffraction and interference effects are significant and one must take them into account.
- For *high frequency* situations, we can approximate or even ignore diffraction, but probably still care about polarization and interference.
- For *very high frequency* situations (like rendering) we usually ignore all interference, diffraction, and polarization.

- Geometric Optics applies to high and very high frequency cases, but can be extended to handle low and medium frequency cases by adding diffraction.
Geometric Optics

• Formal geometric optics includes
  • Ray paths (reflection, refraction, etc.)
  • Electromagnetic field vectors
  • Polarization
  • Interference
  • Amplitude function along rays
  • Wavefront curvatures & caustic points
  • Frequency (and/or spectral distributions)

• Basically, it includes everything but diffraction
Wavefront Curvature along a Ray

- Plane wave
- Cylinder wave
- Spherical wave
- Astigmatic wave
Caustic Points

• Zero, one, or two (non-infinite) caustic points along a ray

• Plane wave: both at infinity
• Cylinder wave: one finite, one infinite
• Sphere wave: both at same finite point
• Astigmatic wave: both at different finite points
Astigmatic Rays (General Case)

- Amplitude of electric field along a ray (relative to amplitude at $s = 0$):
  \[ A(s) = \sqrt{\frac{\rho_1 \rho_2}{(\rho_1+s)(\rho_2+s)}} \]

- Principal radii of curvature: $p_1$ & $p_2$
- Note that $A(0) = 1$
- Also note that $s = 0$ doesn’t have to be at the ray origin (in case $A$ goes to infinity at the origin)
Reflected & Refracted Wavefronts

• When a curved wavefront reflects (or refracts) off of a curved surface, we get a new curved wavefront

• “Local Curvature of Wavefronts in an Optical System”, Kneisly, 1964
Reflected & Refracted Wavefront Curvature

• Reflected wavefront curvature:
  \[ L_{out} = L_{in} - 2 \cos \theta_i L_{surf} \]
  \[ M_{out} = -M_{in} + 2M_{surf} \]
  \[ N_{out} = N_{in} - 2N_{surf}/\cos \theta_i \]

• Refracted wavefront curvature:
  \[ L_{out} = \mu L_{in} + \gamma L_{surf} \]
  \[ M_{out} = (\cos \theta_i / \cos \theta_t)(\mu M_{in} + (\gamma / \cos \theta_i)M_{surf}) \]
  \[ N_{out} = (\cos^2 \theta_i / \cos^2 \theta_t)(\mu N_{in} + (\gamma / \cos^2 \theta_i)N_{surf}) \]
  \[ \mu = n_{in}/n_{out} \]
  \[ \gamma = \cos \theta_t - \mu \cos \theta_i \]
Electromagnetic Field of a Ray

• The ray propagates in direction \( \mathbf{k} \)
• The electric field vector \( \mathbf{E} \) oscillates along the ray (red)
• The magnetic field vector \( \mathbf{B} \) of the ray is rotated 90 degrees (blue)
• They are related by:

\[
\mathbf{B} = y \mathbf{k} \times \mathbf{E}
\]

\[
y = \sqrt{\frac{\varepsilon_r \varepsilon_0}{\mu_0}}
\]

• The power of a ray is proportional to the amplitude squared
Frequency & Light Sources

- Monochrome vs. spectrum
- Coherence

- Light Sources
  - Incandescent (thermal / blackbody)
  - Fluorescent
  - LED
  - Laser
  - Molecular emission
Polarization
Complex Numbers

• Recall Euler’s equation:
  \[ e^{i\varphi} = \cos \varphi + i\sin \varphi \]

• \( e^{i\varphi} \) will have a magnitude of 1.0, for any value of \( \varphi \)

• Therefore, when we multiply some complex number by \( e^{i\varphi} \), we don’t change the magnitude, we just rotate it by \( \varphi \)
Sine Waves

• We can use a complex number to represent an angle (phase) and amplitude of a sine wave at some reference point $r = 0$

$$\mathcal{E} = A e^{i\theta}$$

• When we multiply that by $e^{i\varphi}$, we advance the phase while keeping the amplitude $A$ constant

• If we have a wavelength $\lambda$, we can describe a sine wave along line $r$:

$$\text{Real}\{\mathcal{E} e^{ir(2\pi/\lambda)}\}$$
Sound Waves

• Sound waves (in non-viscous fluids) are *longitudinal waves* which means the pressure field oscillates along the 1D line of the ray direction.

• We can use a complex number $\mathcal{E}$ to represent an angle (phase) and amplitude of a wave at some reference point and describe the pressure along the ray as:

$$ p(r) = \text{Real}\{\mathcal{E}e^{-ir(2\pi/\lambda)}\} $$

• Note: the minus sign is a matter of convention.
Transverse Waves

• EM waves are *transverse waves* meaning that the field oscillates in the 2D plane perpendicular to the direction of ray propagation.

• This adds to the complexity of EM waves and introduces the concept of *polarization* to these waves/rays.
Plane Wave Polarizations
Jones Vector

- If we’re working with some fixed wavelength $\lambda$, we have two independent oscillations in two orthogonal axes in the ray plane.
- If we choose a frame of reference in this transverse plane (such as horizontal/vertical axes), we can represent the amplitude and phase of this oscillation with two complex numbers.
- We group these into a 2D complex vector called a *Jones Vector*.

$$\mathbf{E} = \begin{bmatrix} \mathcal{E}_x \\ \mathcal{E}_y \end{bmatrix} = \begin{bmatrix} A_x e^{-i\theta_x} \\ A_y e^{-i\theta_y} \end{bmatrix}$$
Jones Vectors (Linear Polarization)

\[ \mathbf{\varepsilon} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \]

\[ \mathbf{\varepsilon} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \]

\[ \mathbf{\varepsilon} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \]

\[ \mathbf{\varepsilon} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \]
Jones Vectors (Circular Polarization)

\[ \mathbf{E} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -i \end{bmatrix} \]

Right circular polarization

\[ \mathbf{E} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ i \end{bmatrix} \]

Left circular polarization
When a reflection, refraction, or other scattering event occurs, a new ray is produced with a new polarization state.

If we assume a linear relationship, we can represent the change in polarity with a 2x2 complex matrix called a Jones matrix:

\[ \mathbf{E}' = \mathbf{J} \cdot \mathbf{E} = \begin{bmatrix} J_{xx} & J_{xy} \\ J_{yx} & J_{yy} \end{bmatrix} \cdot \mathbf{E} \]
Reflection & Refraction

• To model reflection & refraction, we need to define local frames for each ray. Note that all rays lie in the same plane (normal to $\mathbf{s}$)
Polarized Fresnel Reflection

\[ n = n_t/n_i \]

\[ r_s = \frac{\cos \theta_i - \sqrt{n^2 - \sin^2 \theta_i}}{\cos \theta_i + \sqrt{n^2 - \sin^2 \theta_i}} \]

\[ r_p = \frac{n^2 \cos \theta_i - \sqrt{n^2 - \sin^2 \theta_i}}{n^2 \cos \theta_i + \sqrt{n^2 - \sin^2 \theta_i}} \]

\[ \mathbf{\epsilon}' = \begin{bmatrix} r_s & 0 \\ 0 & r_p \end{bmatrix} \cdot \mathbf{\epsilon} \]
Polarized Fresnel Refraction

\[ n = \frac{n_t}{n_i} \]

\[ t_s = \frac{2 \cos \theta_i}{\cos \theta_i + \sqrt{n^2 - \sin^2 \theta_i}} \]

\[ t_p = \frac{2n \cos \theta_i}{n^2 \cos \theta_i + \sqrt{n^2 - \sin^2 \theta_i}} \]

\[ \mathbf{E}' = \begin{bmatrix} t_s & 0 \\ 0 & t_p \end{bmatrix} \cdot \mathbf{E} \]
Polarization Vector

• Jones vectors & matrices work fine if you are working in these canonical frames of reference
• When we have many rays interacting in 3D space, we can use the 3D polarization vector method
• Similar to the 2D Jones vector, except done in a single global frame, so that every ray does not require its own local frame

\[ \mathbf{E} = \begin{bmatrix} \mathcal{E}_x \\ \mathcal{E}_y \\ \mathcal{E}_z \end{bmatrix} \]

• Note \( \mathcal{E}_x, \mathcal{E}_y, \) and \( \mathcal{E}_z \) are complex numbers
Polarization Vector

\[ \mathbf{E}(r, t) = \text{Real}\{\mathbf{\mathcal{E}}e^{i(krn+\phi-\omega t)}\} \]

- \( \mathbf{E} \): Electric field vector
- \( \mathbf{\mathcal{E}} \): Polarization vector
- \( k \): Wave number \((2\pi/\lambda)\)
- \( r \): Distance along ray
- \( n \): Index of refraction
- \( \phi \): Phase at ray origin
- \( \omega \): Frequency
- \( t \): Time
Reflection and Refraction

• For a reflected or refracted ray, we compute the polarization matrix $\mathbf{P}$ and use it to compute the new polarization vector $\mathbf{E}'$

$$\mathbf{E}' = \mathbf{P} \cdot \mathbf{E}$$

• $\mathbf{P}$ incorporates the change of direction, amplitude, and polarization based on the Fresnel equations and Snell’s law
Reflection Polarization Matrix

\[ \mathbf{P}_{\text{fresnel}} = \mathbf{O}_{\text{out}} \cdot \mathbf{R}_{\text{fresnel}} \cdot \mathbf{O}_{\text{in}}^{-1} \]

\[ \mathbf{P}_{\text{fresnel}} = \begin{bmatrix} s_x & p_{out_x} & k_{out_x} \\ s_y & p_{out_y} & k_{out_y} \\ s_z & p_{out_z} & k_{out_z} \end{bmatrix} \cdot \begin{bmatrix} r_s & 0 & 0 \\ 0 & r_p & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} s_x & s_y & s_z \\ p_{in_x} & p_{in_y} & p_{in_z} \\ k_{in_x} & k_{in_y} & k_{in_z} \end{bmatrix} \]
Polarization Matrix

• A polarization matrix $\mathbf{P}_i$ can be built for any particular interaction along a ray path, such as reflections, refractions, diffractions, absorption, birefringence, volumetric scattering, interactions with thin coatings, filters, polarizers, etc.

• These can be chained together along a ray path to model a sequence of interactions

$$\mathbf{E}' = \mathbf{P}_n \cdots \mathbf{P}_2 \cdot \mathbf{P}_1 \cdot \mathbf{E}$$
Polarization Effects

• Specular reflection
• Filters
• Birefringence
• Rayleigh scattering
Coherent & Incoherent Light

• Our discussion on polarization has assumed that we are dealing with coherent light that maintains a steady frequency over time

• This may be a valid assumption when working with lasers, radar, radio waves, and many other situations

• Many real-world light sources, however, are less consistent and produce incoherent light

• Incoherent sources (like incandescent, fluorescent, and LED lights) produce waves with shifting phase offsets and don’t produce more than a few wavelengths of coherency

• There is another method for modeling polarization that uses Stokes vectors and Mueller matrices to account for incoherent polarization
Interference

• If multiple rays pass through a point, we can evaluate the electric field vector from each ray and simply sum them up to get the final result
• This will result in interference (possibly constructive or destructive...)
• We can also do this with the complex polarization vectors
Luneberg-Kline Geometric Optics

• Electric field (vector) along a ray:

\[ \mathbf{E}(s) = \text{Real}\left\{\mathcal{E}(0) \sqrt{\frac{\rho_1 \rho_2}{(\rho_1+s)(\rho_2+s)}} e^{-iks} e^{i(n-m)\pi/2}\right\} \]

Wave number: \( k = \frac{2\pi}{\lambda} \)

Principal radii of curvature: \( p_1 \) & \( p_2 \)

[Kline, 1951]
Geometric Optics
Geometric Optics

• Luneberg-Kline geometric optics includes
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• Basically, it includes everything but diffraction
Uniform Theory of Diffraction
Uniform Theory of Diffraction (UTD)

• Geometric Theory of Diffraction [Keller, 1962]
• Uniform Theory of Diffraction [Kouyoumjian, Pathak, 1974]

• Additional field to existing geometric optics field
• Works for medium to high frequency cases

• Present-day applications:
  • Antenna modeling
  • RF coverage analysis
  • Sound propagation

• Related techniques:
  • BTM diffraction (Boit-Tolstoy-Medvin)
  • Method of Equivalent Currents
  • Physical optics
2D Wedge Diffraction

- Reflection zone
- Reflection boundary
- Direct illuminated zone
- Shadow boundary
- Shadow zone
- Source
Uniform Theory of Diffraction

\[ D = D_1 + D_2 + R(D_3 + D_4) \]

\[ D_1 = \frac{-e^{-i\pi/4}}{2n\sqrt{2\pi k}} \frac{\cot \left( \frac{\pi + (\varphi - \varphi')}{2n} \right) F \left( kL a^+ (\varphi - \varphi') \right)}{\sin \beta_0} \]

\[ D_2 = \frac{-e^{-i\pi/4}}{2n\sqrt{2\pi k}} \frac{\cot \left( \frac{\pi - (\varphi - \varphi')}{2n} \right) F \left( kL a^- (\varphi - \varphi') \right)}{\sin \beta_0} \]

\[ D_3 = \frac{-e^{-i\pi/4}}{2n\sqrt{2\pi k}} \frac{\cot \left( \frac{\pi + (\varphi + \varphi')}{2n} \right) F \left( kL^{r_n} a^+(\varphi + \varphi') \right)}{\sin \beta_0} \]

\[ D_4 = \frac{-e^{-i\pi/4}}{2n\sqrt{2\pi k}} \frac{\cot \left( \frac{\pi - (\varphi + \varphi')}{2n} \right) F \left( kL^{r_o} a^- (\varphi + \varphi') \right)}{\sin \beta_0} \]

\[ F(x) = 2i\sqrt{x} e^{ix} \int_{\sqrt{x}}^{\infty} e^{-ix^2} du \]
Incident Field
Reflected Field
Incident + Reflected Field
Incident + Reflected + UTD Field
UTD Field
Path Analysis

Solid object

Source

Target

Direct path
Reflected path
Diffracted path
3D Edge Diffraction
3D Corner Diffraction
3D UTD

NOTES:
- Diffraction
- Polarization
- Interference
- Spectral color
- Fresnel reflection
- No noise
- 20 FPS
References
Thank You!