Importance sampling (continued^2)
Area light sampling

- two ways to sample a light source

1. randomly pick a point on the light source

2. randomly sample a direction from the shading point

**quiz**: what is the integral we are solving here?
Area light sampling

- two ways to sample a light source

1. randomly pick a point on the light source

2. randomly sample a direction from the shading point

\[
\int f_r(\omega_v, \omega_l) \frac{\left| \omega_l \cdot n_p \right|}{\left| \omega_l \cdot n_{p'} \right|} \frac{\| p' - p \|^2}{\text{vis}(p, p')Ldp'}
\]

**quiz:** what are the pros and cons?
Area integral vs solid angle integral

\[
\int f_r(\omega_v, \omega_l) \frac{|\omega_l \cdot n_p|}{\| p' - p \|^2} \frac{|\omega_l \cdot n_{p'}|}{\text{vis}(p, p')Ldp'} = \int ?d\omega_l
\]
Area integral vs solid angle integral

\[
\int f_r(\omega_v, \omega_l) \frac{\omega_l \cdot n_p}{\|p' - p\|^2} \text{vis}(p, p')Ldp' = \int f_r(\omega_v, \omega_l) |\omega_l \cdot n_p| L(\omega_l)d\omega_l
\]
Area integral vs solid angle integral

\[
\int f_r(\omega_v, \omega_l) \frac{\mid \omega_l \cdot n_p \mid \mid \omega_l \cdot n_{p'} \mid}{\mid p' - p \mid^2} \text{vis}(p, p')Ldp' = \int f_r(\omega_v, \omega_l) \mid \omega_l \cdot n_p \mid L(\omega_l)d\omega_l
\]

want to importance sample this!
Importance sampling is crucial for noise reduction.
Cosine-weighted hemisphere sampling

\[ F \int f_r(\omega_v, \omega_l) \left| \omega_l \cdot n_p \right| L(\omega_l) d\omega_l \]

want to sample: \( \omega \sim p(\omega) \propto \cos \theta = \omega_z \)
Strategy: convert Cartesian coordinates to spherical coordinates

\[ (u_1, u_2) \]

\[
\begin{align*}
x &= \cos \phi \sin \theta \\
y &= \sin \phi \sin \theta \\
z &= \cos \theta
\end{align*}
\]

Jacobian = \( \sin \theta \)

\( (x, y, z) \)
Next: construct a mapping between a square and the spherical coordinates

\[ p(\theta, \phi) \propto \sin \theta \cos \theta \quad \theta \in \left[0, \frac{\pi}{2}\right] \quad \phi \in [0, 2\pi] \]

sample \( \theta \sim p(\theta) \) first

sample \( \phi \sim p(\phi | \theta) \)
Next: construct a mapping between a square and the spherical coordinates

\[ p(\theta, \phi) \propto \sin \theta \cos \theta \quad \theta \in \left[0, \frac{\pi}{2}\right] \quad \phi \in [0, 2\pi] \quad p(\theta, \phi) = \frac{\sin \theta \cos \theta}{\pi} \]

Sample \( \theta \sim p(\theta) \) first

\[ p(\theta) = \int_0^{2\pi} p(\theta, \phi) d\phi = 2 \cos \theta \sin \theta = \sin 2\theta \]

Sample \( \phi \sim p(\phi | \theta) \)

\[ p(\phi | \theta) = \frac{p(\theta, \phi)}{p(\theta)} = \frac{1}{2\pi} \]
Next: construct a mapping between a square and the spherical coordinates

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sample \( \phi \sim p(\phi | \theta) \)

\[ p(\phi | \theta) = \frac{p(\theta, \phi)}{p(\theta)} = \frac{1}{2\pi} \]

\[ F^{-1}(\theta) = -\frac{1}{2} \cos 2\theta + \frac{1}{2} \]

\[ F^{-1}(\phi | \theta) = \frac{\phi}{2\pi} \]
Next: construct a mapping between a square and the spherical coordinates

\[ p(\theta, \phi) \propto \sin \theta \cos \theta \quad \theta \in \left[ 0, \frac{\pi}{2} \right] \quad \phi \in [0, 2\pi] \quad p(\theta, \phi) = \frac{\sin \theta \cos \theta}{\pi} \]

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sample \( \phi \sim p(\phi | \theta) \)

\[ p(\phi | \theta) = \frac{p(\theta, \phi)}{p(\theta)} = \frac{1}{2\pi} \]

\[ F^{-1}(\theta) = -\frac{1}{2} \cos 2\theta + \frac{1}{2} \quad \theta = \frac{1}{2} \cos^{-1}(1 - 2u_1) \]

\[ F^{-1}(\phi | \theta) = \frac{\phi}{2\pi} \quad \phi = 2\pi u_2 \]

\( (u_1, u_2) \quad (\theta, \phi) \quad (x, y, z) \)
Alternative: Malley's method

- uniformly sample a point on a disk
- project the point on the hemisphere

\[
\left| \frac{d\omega'}{du} \right| = 1 / \left| \frac{du}{d\omega'} \right| = \pi / |\omega' \cdot n|.
\]

generic intuition: the cosine term $|\omega' \cdot n|$ is the ratio between an area on hemisphere and an area on surface so the Jacobian of the mapping is that ratio divided by the area of a unit disk.
Recall: area light sampling

- two ways to sample a light source
  1. randomly pick a point on the light source
  2. randomly sample a direction from the shading point

quiz: what is the integral we are solving here?
Recall: area light sampling

- two ways to sample a light source

1. randomly pick a point on the light source
2. randomly sample a direction from the shading point

\[
\int f_r(\omega_v, \omega_l) \mid \omega_l \cdot n \mid L(\omega_l)d\omega_l
\]
Importance sampling microfacet BRDFs

given a viewing direction $\omega_v$, how do we sample $\omega_l \sim p(\omega_l) \propto f_r(\omega_v, \omega_l) |\omega_l \cdot n|$?

$$\int f_r(\omega_v, \omega_l) |\omega_l \cdot n| L(\omega_l) d\omega_l$$
Importance sampling microfacet BRDFs

given a viewing direction $\omega_v$, how do we sample $\omega_l \sim p(\omega_l) \propto f_r(\omega_v, \omega_l) |\omega_l \cdot n|$

we usually can’t do this perfectly!
don’t know how to integrate/invert $p$ in general

$$\int f_r(\omega_v, \omega_l) |\omega_l \cdot n| L(\omega_l)d\omega_l$$
Recall: microfacet BRDFs

\[ f_r(\omega_v, \omega_l) = \frac{D(h)G(\omega_v, \omega_l, h)F(\omega_v, h)}{4 \left| \omega_v \cdot n \right| \left| \omega_l \cdot n \right|} \]

\[ \int f_r(\omega_v, \omega_l) \left| \omega_l \cdot n \right| L(\omega_l) d\omega_l \]

**quiz 1:** what is \( h \)?

**quiz 2:** what are \( D, G, \) and \( F \)?

**quiz 3:** for highly specular BRDFs, which term has the largest variation?
Recall: microfacet BRDFs

\[ f_r(\omega_v, \omega_l) = \frac{D(h)G(\omega_v, \omega_l, h)F(\omega_v, h)}{4 |\omega_v \cdot n||\omega_l \cdot n|} \]

\[ \int f_r(\omega_v, \omega_l) |\omega_l \cdot n| L(\omega_l) d\omega_l \]

the cosine term cancels out!

**quiz 1:** what is \( h \)?

**quiz 2:** what are \( D, G, \) and \( F \)?

**quiz 3:** for highly specular BRDFs, which term has the largest variation?
Recall: microfacet BRDFs

\[ f_r(\omega_v, \omega_l) = \frac{D(h)G(\omega_v, \omega_l, h)F(\omega_v, h)}{4 |\omega_v \cdot n| |\omega_l \cdot n|} \]

usually importance sample the NDF which has the largest variation

\[ \int f_r(\omega_v, \omega_l) |\omega_l \cdot n| L(\omega_l) d\omega_l \]

quiz 1: what is \( h \)?

quiz 2: what are \( D \), \( G \), and \( F \)?

quiz 3: for highly specular BRDFs, which term has the largest variation?
Idea: sampling micronormal based on the NDF $D$
Idea: sampling micronormal based on the NDF $D$
Importance sampling Blinn-Phong NDF

\[ D(h) = \frac{s + 2}{2\pi} (n \cdot h)^s \]

want to sample \( h \sim p(h) \propto D(h) \)
Importance sampling Blinn-Phong NDF

\[ p(\theta_h, \phi_h) \propto \sin \theta_h \left( \cos \theta_h \right)^s \quad \theta_h \in \left[ 0, \frac{\pi}{2} \right] \quad \phi_h \in [0, 2\pi] \]

sample \( \theta_h \sim p(\theta_h) \) first

sample \( \phi_h \sim p(\phi_h | \theta_h) \)
Importance sampling Blinn-Phong NDF

\[ p(\theta_h, \phi_h) \propto \sin \theta_h (\cos \theta_h)^s \quad \theta_h \in \left[ 0, \frac{\pi}{2} \right] \quad \phi_h \in [0, 2\pi] \quad p(\theta_h, \phi_h) = \frac{s + 1}{2\pi} \sin \theta_h (\cos \theta_h)^s \]

sample \( \theta_h \sim p(\theta_h) \) first

\[ p(\theta_h) = \int_0^{2\pi} p(\theta_h, \phi_h) d\phi_h = (n + 1)\sin \theta_h (\cos \theta_h)^s \]

sample \( \phi_h \sim p(\phi_h | \theta_h) \)

\[ p(\phi_h | \theta_h) = \frac{p(\theta_h, \phi_h)}{p(\theta_h)} = \frac{1}{2\pi} \]
Importance sampling Blinn-Phong NDF

\[ p(\theta_h, \phi_h) \propto \sin \theta_h (\cos \theta_h)^s \quad \theta_h \in \left[ 0, \frac{\pi}{2} \right] \quad \phi_h \in [0,2\pi] \quad p(\theta_h, \phi_h) = \frac{s + 1}{2\pi} \sin \theta_h (\cos \theta_h)^s \]

Sample \( \theta_h \sim p(\theta_h) \) first

Sample \( \phi_h \sim p(\phi_h | \theta_h) \)

\[ F^{-1}(\theta_h) = \int_0^{\theta_h} p(\theta_h) d\theta_h = 1 - (\cos \theta_h)^{n+1} \]

\[ F^{-1}(\phi_h | \theta_h) = \int_0^{\phi_h} p(\phi_h | \theta_h) d\phi_h = \frac{\phi_h}{2\pi} \]
Importance sampling Blinn-Phong NDF

\[ p(\theta_h, \phi_h) \propto \sin \theta_h \left( \cos \theta_h \right)^s \quad \theta_h \in \left[ 0, \frac{\pi}{2} \right] \quad \phi_h \in [0, 2\pi] \quad p(\theta_h, \phi_h) = \frac{s + 1}{2\pi} \sin \theta_h \left( \cos \theta_h \right)^s \]

sample \( \theta_h \sim p(\theta_h) \) first

sample \( \phi_h \sim p(\phi_h | \theta_h) \)

\[ F^{-1}(\theta_h) = \int_0^{\theta_h} p(\theta_h) \, d\theta_h = 1 - \left( \cos \theta_h \right)^{n+1} \]

\[ F^{-1}(\phi_h | \theta_h) = \int_0^{\phi_h} p(\phi_h | \theta_h) \, d\phi_h = \frac{\phi_h}{2\pi} \]

\[ p(\theta_h) = \int_0^{2\pi} p(\theta_h, \phi_h) \, d\phi_h = (n + 1) \sin \theta_h \left( \cos \theta_h \right)^s \]

\[ p(\phi_h | \theta_h) = \frac{p(\theta_h, \phi_h)}{p(\theta_h)} = \frac{1}{2\pi} \]

\[ \theta_h = \cos^{-1} \left( u_1^{\frac{1}{n+1}} \right) \]

\[ \phi_h = \frac{u_2}{2\pi} \]
Recall: our goal is to sample $\omega_v$, not $h$

**quiz:** given $\omega_v$ and $h$, what is $\omega_l$?
Recall: our goal is to sample $\omega_l$, not $h$

**quiz:** given $\omega_v$ and $h$, what is $\omega_l$?

$$\omega_l = \omega_v - \left(2\omega_v \cdot h\right) h$$

(assuming $\omega_h$ is pointing towards the surface)
Combining the two steps: what is the final PDF?

\[ (u_1, u_2) \xrightarrow{F_1} h \sim p(h) \propto D(h) \xrightarrow{F_2} \omega_l = \text{reflect}(\omega_v, h) \]
Combining the two steps: what is the final PDF?

$$\omega_l = \text{reflect}(\omega_v, h)$$

$$h \sim p(h) \propto D(h)$$

$$\text{final PDF} = p(\omega_l) = \left| \frac{du}{d\omega_l} \right| = \left| \frac{du}{dh} \right| \left| \frac{dh}{d\omega_l} \right| = p(h) \frac{1}{4(h \cdot \omega_l)} = p(h) \frac{1}{4(h \cdot \omega_v)}$$
Recall: microfacet BRDFs

\[
f_r(\omega_v, \omega_l) \mid \omega_l \cdot n \mid = \frac{D(h)G(\omega_v, \omega_l, h)F(\omega_v, h)}{4 \mid \omega_v \cdot n \mid}
\]

\[
p(\omega_l) = p(h) \frac{1}{4(h \cdot \omega_v)} \propto \frac{\mid h \cdot n \mid^s}{4(h \cdot \omega_v)} = \frac{D(h)}{4(h \cdot \omega_v)}
\]

note: the equations provided in homework 4 importance sample \(D(h)h \cdot n\) instead of \(D(h)\)
Recall: microfacet BRDFs

\[ f_r(\omega_v, \omega_l) \left| \omega_l \cdot n \right| = \frac{D(h)G(\omega_v, \omega_l, h)F(\omega_v, h)}{4\left| \omega_v \cdot n \right|} \]

\[ p(\omega_l) = p(h) \frac{1}{4(h \cdot \omega_v)} \propto \frac{\left| h \cdot n \right|^s}{4(h \cdot \omega_v)} = \frac{D(h)}{4(h \cdot \omega_v)} \]

note: the equations provided in homework 4 importance sample \( D(h)h \cdot n \) instead of \( D(h) \)
Detail: normalization of NDF vs PDF

\[ p(\theta_h, \phi_h) = \frac{s + 1}{2\pi} \sin \theta_h (\cos \theta_h)^s \]

\[ p(h) = \frac{s + 1}{2\pi} (n \cdot h)^s \]

**quiz:** why are these two different?
Detail: normalization of NDF vs PDF

\[ D(h) = \frac{s + 2}{2\pi} (n \cdot h)^s \]

\[ p(\theta_h, \phi_h) = \frac{s + 1}{2\pi} \sin \theta_h (\cos \theta_h)^s \]

\[ p(h) = \frac{s + 1}{2\pi} (n \cdot h)^s \]

NDFs are normalized over projection area, i.e.

\[ \int D(h) |n \cdot h| \, dh = 1 \]

“How is the NDF really defined?” by Nathan Reed

https://www.reedbeta.com/blog/hows-the-ndf-really-defined/
Sampling a cone

want to sample \((\theta, \phi) \sim p(\theta, \phi) \propto \frac{\sin \theta}{2\pi}, \theta \in [0, \theta_{\text{max}}]\)
Sampling a cone

want to sample \((\theta, \phi) \sim p(\theta, \phi) \propto \frac{\sin \theta}{2\pi}, \theta \in [0, \theta_{\text{max}}]\)

\[
p(\theta) = \frac{\sin \theta}{1 - \cos \theta_{\text{max}}}
\]

\[
p(\phi \mid \theta) = \frac{1}{2\pi}
\]
Sampling a cone

want to sample \((\theta, \phi) \sim p(\theta, \phi) \propto \frac{\sin \theta}{2\pi}, \theta \in [0, \theta_{\text{max}}]\)

\[p(\theta) = \frac{\sin \theta}{1 - \cos \theta_{\text{max}}}\]

\[p(\phi | \theta) = \frac{1}{2\pi}\]

\[\theta = (1 - u_1) + u_1 \cos \theta_{\text{max}}\]

\[\phi = 2u_2 \pi\]
Sampling a discrete distribution

given a probability mass function, how do we sample from it?
Sampling a discrete distribution

given a probability mass function, how do we sample from it?

\[ \int_0^x f(x') \, dx' \]

treat it as a piecewise constant function
Sampling a discrete distribution

\[ p = [\frac{1}{7}, \frac{3}{7}, \frac{1}{7}, \frac{2}{7}] \]
Sampling a discrete distribution

\[ p = \left[ \frac{1}{7}, \frac{3}{7}, \frac{1}{7}, \frac{2}{7} \right] \quad \text{integrate} \quad P = \left[ \frac{1}{7}, \frac{4}{7}, \frac{5}{7}, \frac{7}{7} \right] \]

sample \( u = \frac{1}{3} \)
Sampling a discrete distribution

\[ p = \left[ \frac{1}{7}, \frac{3}{7}, \frac{1}{7}, \frac{2}{7} \right] \]
integrate

\[ P = \left[ \frac{1}{7}, \frac{4}{7}, \frac{5}{7}, \frac{7}{7} \right] \]

sample \( u = \frac{1}{3} \)

invert: find first index \( i \) s.t. \( P_i \geq u \)
Next: distributed rendering