Importance sampling

UCSD CSE 168
Rendering
Tzu-Mao Li
Discussion on Wednesday

• options:
  • modern c++ (parallelism, lambda functions, optional, variant, etc)?
  • rendering spherical light sources?
  • proposals?
Last time: Monte Carlo integration

\[ \int f(x)dx \approx \frac{1}{N} \sum_{i=1}^{N} \frac{f(x_i)}{p(x_i)} \]
Last time: Monte Carlo integration

\[ \int f(x) \, dx \approx \frac{1}{N} \sum_{i=1}^{N} \frac{f(x_i)}{p(x_i)} \]

**quiz 1**: what is the expectation of the estimator?
Last time: Monte Carlo integration

\[ \int f(x)dx \approx \frac{1}{N} \sum_{i=1}^{N} \frac{f(x_i)}{p(x_i)} \]

**quiz 1:** what is the expectation of the estimator?
**quiz 2:** what is a good probability density \( p \)?
Last time: Monte Carlo integration

\[ \int f(x) dx \approx \frac{1}{N} \sum_{i=1}^{N} \frac{f(x_i)}{p(x_i)} \]

**quiz 1**: what is the expectation of the estimator?
**quiz 2**: what is a good probability density \( p \)?
**quiz 3**: how do we generate \( x_i \sim p(x_i) \)?
Importance sampling:
choose $p$ to be similar to $f$

$$\int f(x) dx \approx \frac{1}{N} \sum_{i=1}^{N} \frac{f(x_i)}{p(x_i)}$$

**quiz:** what if $p \propto f$?
Example: importance sampling a product

\[ \int f_r(x) L(x) \, dx \approx \frac{1}{N} \sum_{i=1}^{N} \frac{f_r(x_i) L(x_i)}{p(x_i)} = \frac{1}{N} \sum_{i=1}^{N} \frac{L(x_i)}{c} \]

\[ p(x_i) \propto f_r(x_i) \]
How do we sample from a probability density?

\[ x \sim p(x) \]
Idea: map from uniform distribution
Sampling from uniform distribution:
pseudo random numbers
Sampling from uniform distribution: pseudo random numbers

Linear Congruential Generator (LCG)

\[ u_0 = \text{seed} \]

\[ u_{i+1} = (au_i + b) \mod N \]

choose \( a, b \) s.t. \( u_0 \) to \( u_{N-1} \) covers all integers in \([0, N-1]\) exactly once

“PCG: A Family of Better Random Number Generators”, Melissa O’Neil
https://www.youtube.com/watch?v=45Oet5qJlms
Sampling from uniform distribution: pseudo random numbers

Linear Congruential Generator (LCG)

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for \( N = 2^{32} - 1 \), let \( a = 16807 \) and \( b = 0 \)

(minstd [Park and Miller 1988])

“PCG: A Family of Better Random Number Generators”, Melissa O’Neil

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A few ways to sample non-uniform distributions

- rejection sampling
- **inverse transform sampling** (most important)
- Metropolis sampling (will cover in CSE 272)
- importance resampling (will cover in CSE 272)
A few ways to sample non-uniform distributions

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Uniformly sampling a disk using rejection sampling
Rejection sampling: general algorithm

want to sample $x \sim p(x)$
Rejection sampling: general algorithm

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- find an upper bound $M$ s.t. $p(x) \leq M$
want to sample $x \sim p(x)$

- find an upper bound $M$ s.t. $p(x) \leq M$
- sample $u_x \propto U$
Rejection sampling: general algorithm

want to sample $\mathbf{x} \sim p(\mathbf{x})$

- find an upper bound $M$ s.t. $p(\mathbf{x}) \leq M$
- sample $\mathbf{u}_x \propto U$
- sample $\mathbf{u}_y \propto U(0,M)$
Rejection sampling: general algorithm

want to sample $x \sim p(x)$

• find an upper bound $M$ s.t. $p(x) \leq M$
• sample $u_x \propto U$
• sample $u_y \propto U(0,M)$
• if $p(u_x) > u_y$ accept the sample
  else, reject the sample
• repeat until accept
Rejection sampling: general algorithm

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• repeat until accept

quiz: why does this work?
Issues with rejection sampling

- high rejection rate when the upper bound is loose
- hard to combine with stratified sampling
A few ways to sample non-uniform distributions

- rejection sampling
- **inverse transform sampling** (most important)
- Metropolis sampling (will cover in CSE 272)
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Idea: map from uniform distribution

\[
x_i = F(u_i)
\]

want to sample \( x \sim p(x) \)
Mapping between distribution
= variable substitution in calculus

\[ x = F(u) \]

\[ \int p_u(u)du = \int du = 1 = \int ?dx \]

want to sample \( x \sim p(x) \)
Mapping between distribution = variable substitution in calculus

\[
\int p_u(u) \, du = \int du = 1 = \int \left| \frac{du}{dx} \right| \, dx = \int p(x) \, dx
\]

\[
x = F(u) \quad \quad \quad u = F^{-1}(x)
\]

want to sample \( x \sim p(x) \)
Mapping between distribution = variable substitution in calculus

\[ \int p_u(u) du = \int du = 1 = \int \left| \frac{du}{dx} \right| dx = \int p(x) dx \]

\[ x = F(u) \quad u = F^{-1}(x) \]

goal: find \( u = F^{-1}(x) \) s.t. \( \left| \frac{du}{dx} \right| = p(x) \)

i.e. we want to find the anti-derivative of \( p \) and then invert it!

want to sample \( x \sim p(x) \)
Inverse transform sampling in 1D

- recipe: want to sample $x \sim p(x)$  \hspace{1cm} \textbf{goal: find} $u = F^{-1}(x)$ \text{s.t.} $\left| \frac{du}{dx} \right| = p(x)$
Inverse transform sampling in 1D

• recipe: want to sample $x \sim p(x)$  
  goal: find $u = F^{-1}(x)$ s.t. $\left| \frac{du}{dx} \right| = p(x)$

• find integral $F^{-1}(x) = \int_{-\infty}^{x} p(x)dx$ (this is the Cumulative Distribution Function of $p$!)
Inverse transform sampling in 1D

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- invert $u = F^{-1}(x)$ and obtain $x = F(u)$
Inverse transform sampling in 1D

• recipe:
  - want to sample $x \sim p(x)$
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  - find integral $F^{-1}(x) = \int_{-\infty}^{x} p(x)dx$ (this is the Cumulative Distribution Function of p!)

• invert $u = F^{-1}(x)$ and obtain $x = F(u)$

• let $x_i = F(u_i)$ where $u_i \sim U(0,1)$
Example: power function

(used in Phong/Blinn-Phong)

\[ p(x) \propto x^n \quad x \in [0,1] \]
Example: power function

(used in Phong/Blinn-Phong)

\[ p(x) \propto x^n \quad x \in [0,1] \]

\[ p(x) = (n + 1)x^n \]

\[ F^{-1}(x) = x^{n+1} \]

\[ F(u) = u^{\frac{1}{n+1}} \]

\[ \int_0^x x^n dx = \frac{x^{n+1}}{n+1} \]
Example: exponential function

(used in participating media rendering)

\[ p(x) \propto e^{-ax} \quad x \in [-\infty, \infty] \]
Example: exponential function

(used in participating media rendering)

\[ p(x) \propto e^{-ax} \quad x \in [-\infty, \infty] \]

\[ p(x) = ae^{-ax} \]

\[ F^{-1}(x) = 1 - e^{-ax} \]

\[ F(u) = -\frac{1}{a} \log (1 - x) \]
High-dimensional inverse transform sampling: deal with one dimension at a time

\[ p(x, y) = p(x)p(y | x) \]

first sample x, then sample y given x
Example: uniformly sampling a disk

\[(u_1, u_2) \rightarrow (x, y)\]

used in depth of field rendering and global illumination
Strategy: convert Cartesian coordinates to polar coordinates

\((u_1, u_2)\)  \(\rightarrow\)  \((r, \phi)\)  \(\rightarrow\)  \((x, y)\)
Uniform sampling of a unit disk

- incorrect approach:
  - uniformly pick a distance $r$ from origin
  - uniformly pick an angle $\phi$

why is this wrong?
Uniform sampling of a unit disk

- incorrect approach:
  - uniformly pick a distance $r$ from origin
  - uniformly pick an angle $\phi$

why is this wrong?

“inner” circles have less area compared to “outer” circles!
Strategy: convert Cartesian coordinates to polar coordinates

\[ x = r \cos \phi \]
\[ y = r \sin \phi \]

\[
\iint dxdy = \int_0^{2\pi} \int_0^1 rdrd\phi
\]

need to account for the Jacobian \( r \)
Next: construct a mapping between a square and the polar coordinates

\[ p(r, \phi) \propto r \quad r \in [0,1] \quad \phi \in [0,2\pi] \]

sample \( r \sim p(r) \) first

sample \( \phi \sim p(\phi | r) \)
Next: construct a mapping between a square and the polar coordinates

\[ p(r, \phi) \propto r \quad r \in [0,1] \quad \phi \in [0,2\pi] \quad p(r, \phi) = \frac{r}{\pi} \]

sample \( r \sim p(r) \) first

\[ p(r) = \int_0^{2\pi} p(r, \phi) d\phi = 2r \]

sample \( \phi \sim p(\phi \mid r) \)

\[ p(\phi \mid r) = \frac{p(\phi, r)}{p(r)} = \frac{1}{2\pi} \]

\[ p(r) = \int \left( \phi \right)_{\phi \sim p(\phi \mid r)} d\phi = \frac{r}{2\pi} \]

\[ p(\phi) = \int \left( r \right)_{r \sim p(r)} d\phi = \frac{1}{2\pi} \]

\[ p(x, y) = \int \left( \phi \right)_{\phi \sim p(\phi \mid r)} \left( r \right)_{r \sim p(r)} d\phi d\phi = \frac{1}{2\pi} \]

\[ (u_1, u_2) \rightarrow (r, \phi) \rightarrow (x, y) \]
Next: construct a mapping between a square and the polar coordinates

\[ p(r, \phi) \propto r \quad r \in [0,1] \quad \phi \in [0,2\pi] \quad p(r, \phi) = \frac{r}{\pi} \]

sample \( r \sim p(r) \) first \[ p(r) = \int_0^{2\pi} p(r, \phi) d\phi = 2r \quad F^{-1}(r) = r^2 \]

sample \( \phi \sim p(\phi \mid r) \) \[ p(\phi \mid r) = \frac{p(\phi, r)}{p(r)} = \frac{1}{2\pi} \quad F^{-1}(\phi \mid r) = \frac{\phi}{2\pi} \]
Next: construct a mapping between a square and the polar coordinates

\[ p(r, \phi) \propto r \quad r \in [0,1] \quad \phi \in [0,2\pi] \quad p(r, \phi) = \frac{r}{\pi} \]

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\[ p(\phi \mid r) = \frac{p(\phi, r)}{p(r)} = \frac{1}{2\pi} \]

\[ F^{-1}(\phi \mid r) = \frac{\phi}{2\pi} \]

\[ r = \sqrt{u_1} \]

\[ \phi = 2\pi u_2 \]
Example: uniform sampling of a disk

\begin{align*}
r &= \sqrt{u_1} \\
\phi &= 2\pi u_2 \\
x &= r \cos \phi \\
y &= r \sin \phi
\end{align*}

used in depth of field rendering and global illumination
WRONG ≠ Equi-Areal

\[ \theta = 2\pi U_1 \]
\[ r = U_2 \]

RIGHT = Equi-Areal

\[ \theta = 2\pi U_1 \]
\[ r = \sqrt{U_2} \]
Shirley-Chiu mapping for disk sampling

• better stratification properties

```c
Vector2 ToUnitDisk(Vector2 O) {
    float phi, r;
    float a = 2*O.x - 1;
    float b = 2*O.y - 1;
    if (a*a > b*b) {
        r = a;
        phi = (PI/4)*(b/a);
    } else {
        r = b;
        phi = (PI/4)*(a/b) + (PI/2);
    }
    return Vector2( r*cos(phi), r*sin(phi) );
}
```

https://psgraphics.blogspot.com/2011/01/improved-code-for-concentric-map.html

A Low Distortion Map Between Disk and Square

Peter Shirley
Kenneth Chiu
Uniform sampling of a triangle

incorrect strategy:
1. uniform sample $b_1$
2. uniform sample $b_2 \sim U(0, 1 - b_1)$

why is this wrong?
Uniform sampling of a triangle

smaller $b_1$ has more area!
Uniform sampling of a triangle

$p(b_1, b_2) \propto 1 \quad b_1 \in [0,1] \quad b_2 \in [0,1-b_1]

sample $b_1 \sim p(b_1)$ first

sample $b_2 \sim p(b_1 | b_2)$
Uniform sampling of a triangle

\[ p(b_1, b_2) = 2 \]
\[ p(b_1, b_2) \propto 1 \quad b_1 \in [0, 1] \quad b_2 \in [0, 1 - b_1] \]

Sample \( b_1 \sim p(b_1) \) first

Sample \( b_2 \sim p(b_1 | b_2) \)

\[ p(b_1) = \int_0^{1-b_1} p(b_1, b_2) \, db_2 = 2(1 - b_1) \]

\[ p(b_2 | b_1) = \frac{p(b_1, b_2)}{p(b_1)} = \frac{1}{1 - b_1} \]
Uniform sampling of a triangle

\[ p(b_1, b_2) = 2 \quad p(b_1, b_2) \propto 1 \quad b_1 \in [0,1] \quad b_2 \in [0,1-b_1] \]

sample \( b_1 \sim p(b_1) \) first

sample \( b_2 \sim p(b_1 | b_2) \)

\[ F^{-1}(b_1) = (1 - b_1)^2 \]

\[ F^{-1}(b_2 | b_1) = \frac{b_2}{1 - b_1} \]
Uniform sampling of a triangle

\[ p(b_1, b_2) = 2 \quad p(b_1, b_2) \propto 1 \quad b_1 \in [0,1] \quad b_2 \in [0,1 - b_1] \]

Sample \( b_1 \sim p(b_1) \) first

Sample \( b_2 \sim p(b_1 | b_2) \)

\[ F^{-1}(b_1) = (1 - b_1)^2 \]

\[ b_1 = 1 - \sqrt{u_1} \]

\[ p(b_1) = \int_0^{1-b_1} p(b_1, b_2) \, db_2 = 2(1 - b_1) \]

\[ p(b_2 | b_1) = \frac{p(b_1, b_2)}{p(b_1)} = \frac{1}{1 - b_1} \]

\[ F^{-1}(b_2 \mid b_1) = \frac{b_2}{1 - b_1} \]

\[ b_2 = \frac{u_2}{\sqrt{u_1}} \]
Uniform sampling of a hemisphere
Strategy: convert Cartesian coordinates to spherical coordinates

\( (u_1, u_2) \)

\[
\begin{align*}
x &= \cos \phi \sin \theta \\
y &= \sin \phi \sin \theta \\
z &= \cos \theta
\end{align*}
\]

Jacobian = \( \sin \theta \)
Next: construct a mapping between a square and the spherical coordinates

\[ p(\theta, \phi) \propto \sin \theta \quad \theta \in \left[0, \frac{\pi}{2}\right] \quad \phi \in [0, 2\pi] \]

sample \( \theta \sim p(\theta) \) first

sample \( \phi \sim p(\phi \mid \theta) \)
Next: construct a mapping between a square and the spherical coordinates

\[ p(\theta, \phi) \propto \sin \theta \quad \theta \in \left[0, \frac{\pi}{2}\right] \quad \phi \in [0, 2\pi] \quad p(\theta, \phi) = \frac{\sin \theta}{2\pi} \]

sample $\theta \sim p(\theta)$ first

\[ p(\theta) = \int_0^{2\pi} p(\theta, \phi) d\phi = \sin \theta \]

sample $\phi \sim p(\phi \mid \theta)$

\[ p(\phi \mid \theta) = \frac{p(\theta, \phi)}{p(\theta)} = \frac{1}{2\pi} \]
Next: construct a mapping between a square and the spherical coordinates

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sample \( \theta \sim p(\theta) \) first

\[ p(\theta) = \int_0^{2\pi} p(\theta, \phi) d\phi = \sin \theta \quad F^{-1}(\theta) = -\cos(\theta) + 1 \]

sample \( \phi \sim p(\phi | \theta) \)

\[ p(\phi | \theta) = \frac{p(\theta, \phi)}{p(\theta)} = \frac{1}{2\pi} \quad F^{-1}(\phi | \theta) = \frac{\phi}{2\pi} \]
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\[ p(\phi | \theta) = \frac{p(\theta, \phi)}{p(\theta)} = \frac{1}{2\pi} \quad F^{-1}(\phi | \theta) = \frac{\phi}{2\pi} \]

\[ \theta = \cos^{-1} \left( 1 - u_1 \right) \]

\[ \phi = 2\pi u_2 \]
Next: more importance sampling!