Day 3 Selected Solutions

Solutions for Exercises 1 and 2

Exercise 1

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$$\begin{aligned} S_1 &= Z \otimes Z \otimes I, \\ S_2 &= I \otimes Z \otimes Z, \\ \overline{X} &= X \otimes X \otimes X \end{aligned}$$

This is the 3 qubit code resistent to X errors.

Exercise 1

Similarly, the 3 qubit code resistent to Z errors will have stabilizers S_3, S_4 that stabilize the space

$$G_3 = \text{linspan}\left(|+++\rangle, |---\rangle\right).$$

Analagously,

$$\begin{aligned} S_3 &= X \otimes X \otimes I, \\ S_4 &= I \otimes X \otimes X, \\ \overline{Z} &= Z \otimes Z \otimes Z \end{aligned}$$

We can see this by noticing $G_3 = (H \otimes H \otimes H)G_2$. So the stabilizer

$$S_3 = (HHH)(ZZI)(HHH) = XXI$$

Exercise 1

This would also work if we worked with logical qubits instead of qubits. So, when we apply the 3 qubit Z-resistent code to the 3 qubit X-resistent code,

$$
\begin{aligned}
S_3 &= \overline{XXI} = XXXXXXIII \\
S_4 &= \overline{IXX} = IIIXXXXXX \\
\overline{\overline{Z}} &= \overline{ZZZ} = ZZZZZZZZZ
\end{aligned}
$$

Exercise 1

So now we need to add the stabilizers for each of the X-resistent codes to get the entire set of stabilizers.

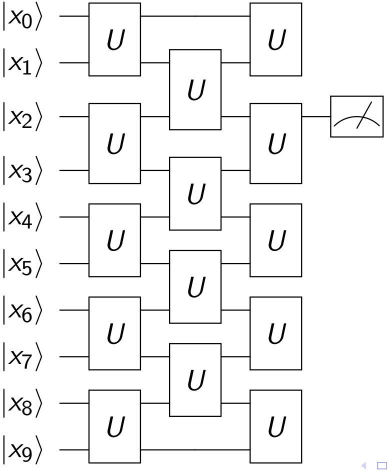
$$\begin{pmatrix} S_1 & = ZZI & III & III \\ S_2 & = IZZ & III & III \\ S_3 & = III & ZZI & III \\ S_4 & = III & IZZ & III \\ S_5 & = III & III & ZZI \\ S_6 & = III & III & IZZ \\ S_7 & = XXX & XXX & III \\ S_8 & = III & XXX & XXX \\ \overline{X} & = XXX & XXX & XXX \\ \overline{Z} & = ZZZ & ZZZ & ZZZ \end{pmatrix}$$

We have a depth-d quantum circuit, and we're interested in a measurement just on one qubit of the final state. How much of the circuit does this depend on? Idea: "backward lightcone".

Exercise 2

Simulating one qubit measurements with constant size quantum circuit

We have a depth-d quantum circuit, and we're interested in a measurement just on one qubit of the final state. How much of the circuit does this depend on? Idea: "backward lightcone".



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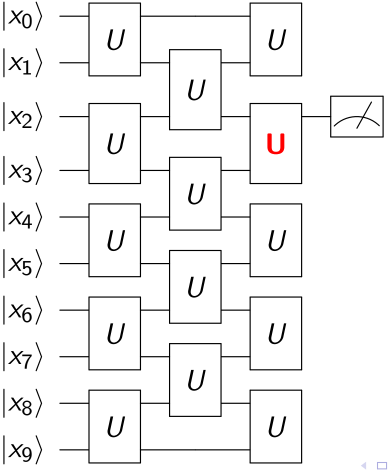

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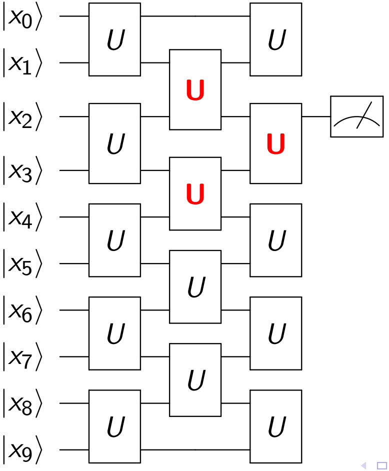

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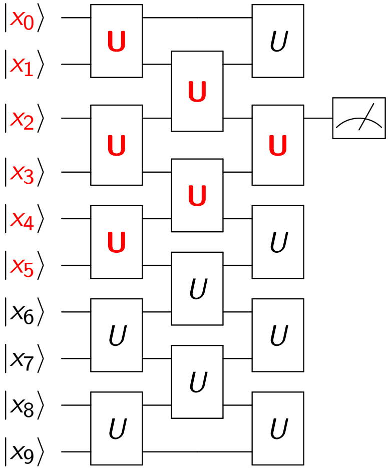

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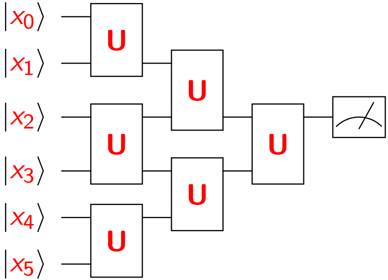

Answer: It depends on at most $D = 2^d$ qubits. This circuit has depth d and acts on D qubits. It has size at most Dd.

The measurement outcome has the same distribution as the same measurement in the original circuit.

Our circuit of size Dd is fully described by a sequence of matrix multiplications, where each matrix has size $2^D \times 2^D$. Notice that the depth of the new circuit $2^D = 2^{2^d}$ is *doubly exponential* in the depth of the original circuit!

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Let U be the unitary describing the circuit. Then we can compute matrix entries of U in constant time using the above.
We can compute the probabilities of the measurement outcome like so:

$$\left\|(\langle 0| \otimes I)C\,|x{\upharpoonright}D\rangle\right\|^2 = \sum_{z\in\{0,1\}^{D-1}} \left|\langle 0z|C|x{\upharpoonright}D\rangle\right|^2 . \qquad (1)$$

Each complex number $\langle 0z|C|x{\upharpoonright}D\rangle$ can be computed in constant time, and we need to compute only a constant number of them. So we can compute the probability of measuring 0.

Exercise 2

Classical constant depth circuits compute the same functions as quantum ones

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Theorem

If quantum circuit of constant depth computes f with probability $\geq \frac{1}{2}$, then there is a constant-depth classical circuit computing f.

Let $|\phi_D\rangle = D^{-1/2}\sum_i |i\rangle |i\rangle$.

$$\langle \phi_D | A \otimes B | \phi_D \rangle = (D^{-1/2} \sum_i \langle i| \langle i|) A \otimes B (D^{-1/2} \sum_j |j\rangle |j\rangle)$$

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&= D^{-1}\sum_{ij} \langle i|A|j\rangle\langle i|B|j\rangle \\
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&= D^{-1} \sum_i \langle i| A B^T |i\rangle = D^{-1} \operatorname{Tr} A B^T .
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$$

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$$ABA^{-1}B^{-1} = A_0 B_0 A_0^{-1} B_0^{-1} \otimes A_1 B_1 A_1^{-1} B_1^{-1}. \qquad (2)$$

Consider the matrices

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\begin{aligned}
C_1 &= Z \otimes I \otimes I \cdots \otimes I \\
C_2 &= X \otimes Z \otimes I \cdots \otimes I \\
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To see this, consider the group commutator term-by-term in the tensor product.

Exercise 7

Constructing the operators

Consider the matrices

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To see this, consider the group commutator term-by-term in the tensor product.
Exactly one group commutator is between an X and Z. The rest are between X and X or between I and something. The commutators of the latter type are all equal to I. The X, Z commutator is equal to $-I$.
Therefore, the whole commutator is $-I$.

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By linearity, this is equal to $\frac{1}{D} \sum_{ij} \operatorname{Tr} u_i v_j C_i C_j^T$.
By our second computational lemma, $\operatorname{Tr} C_i C_j^T = 0$ for $i \neq j$. But also $\frac{1}{D} \operatorname{Tr} C_i C_i^T = 1$.

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Therefore, $\langle \phi_D | C(u) \otimes C(v) | \phi_D \rangle = \sum_i u_i v_i = u \cdot v$.

To conclude: any correlation achieved by inner products of vectors is also achieved by making measurements on a state.