Day 2 Selected Solutions

UCSD Spring School

Exercise 0

$$H_{\text{in}} = |0\rangle\langle 0|_{\text{clock}} \otimes \left(\sum_{i=1}^{n} |1\rangle\langle 1|_i \right)$$

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The projector $|1\rangle\langle 1|_i$ penalizes any state that has support on qubit i of the state register being $|1\rangle_i$ whenever the clock register is in state $|0\rangle_{\text{clock}}$. As this holds for all qubits $1, \ldots, n$ of the state register, then the ground space can be expressed as

$$G(H_{\text{in}}) = \left\{ |\varphi\rangle : |\varphi\rangle = \begin{array}{c} \alpha_0 |0\rangle_{\text{clock}} \otimes |0\rangle_{\text{state}(1..n)}^{\otimes n} \\ + \sum_{t=1}^{T} \alpha_t |t\rangle_{\text{clock}} \otimes |\psi_t\rangle_{\text{state}} \end{array} \right\}.$$

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Similarly,

$$G(H_{\text{out}}) = \left\{ |\varphi\rangle : |\varphi\rangle = \begin{array}{c} \alpha_T\, |T\rangle_{\text{clock}} \otimes |1\rangle_1 \otimes |\eta\rangle_{\text{state}(2..n)} \\ + \sum_{t=0}^{T-1} \alpha_t\, |t\rangle_{\text{clock}} \otimes |\psi_t\rangle_{\text{state}} \end{array} \right\}.$$

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$$H_{\text{prop}}(t) = \frac{1}{2}\big(-|t\rangle\langle t-1|_{\text{clock}} \otimes U_t - |t-1\rangle\langle t|_{\text{clock}} \otimes U_t^{\dagger} + |t\rangle\langle t|_{\text{clock}} \otimes I + |t-1\rangle\langle t-1|_{\text{clock}} \otimes I\big).$$

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$$G(H_{\text{prop}})(t) = \left\{ |\varphi\rangle : |\varphi\rangle = \begin{array}{c} \beta\,|t-1\rangle_{\text{clock}} \otimes |\psi_{t-1}\rangle_{\text{state}} \\ +\beta\,|t\rangle_{\text{clock}} \otimes (U_t\,|\psi_{t-1}\rangle)_{\text{state}} \\ +\sum_{t'\neq t-1,t} \alpha_t\,|t'\rangle_{\text{clock}} \otimes |\psi_{t'}\rangle_{\text{state}} \end{array} \right\}.$$

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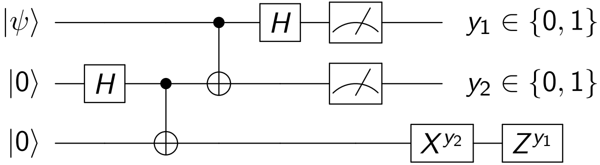
Then consider the intersection of all $G(H_{\text{prop}}(t))$,

$$\begin{aligned} G(H_{\text{prop}}) &= \bigcap_t G(H_{\text{prop}}(t)) \\ &= \left\{ |\varphi\rangle : |\varphi\rangle = \sum_{t=0}^{T} |t\rangle_{\text{clock}} \otimes (U_t \dots U_1\,|\psi_0\rangle)_{\text{state}} \right\} \end{aligned}$$

Exercise 1

By linearity, it suffices to check for $|\psi\rangle = |0\rangle$ and $|\psi\rangle = |1\rangle$.



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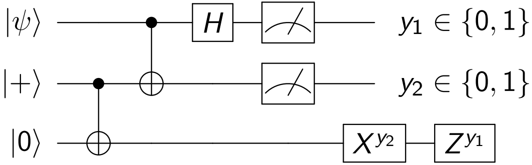

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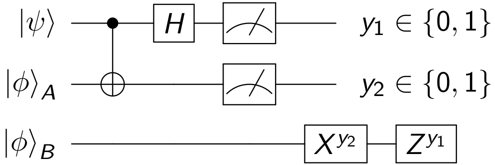

where $|\phi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$.

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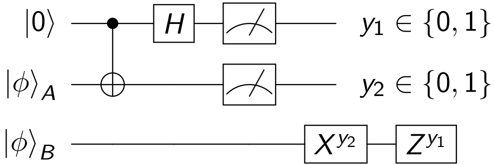
By linearity, it suffices to check for $|\psi\rangle = |0\rangle$ and $|\psi\rangle = |1\rangle$.
We'll set $|\psi\rangle = |0\rangle$ here. Later we'll return and set $|\psi\rangle = |1\rangle$.



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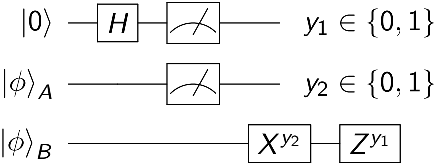
$|\psi\rangle = |0\rangle$.



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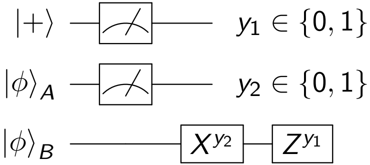

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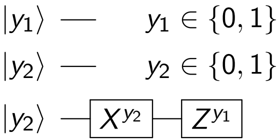

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$$|y_2\rangle \ — \ y_2 \in \{0,1\}$$

$$|0\rangle \ — \boxed{Z^{y_1}}$$

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$$|0\rangle \;——$$

The circuit outputs $|0\rangle$ on the bottom wire, as desired.

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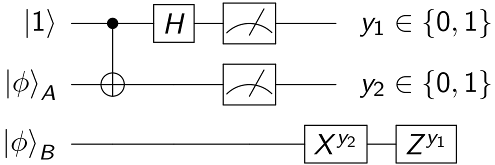
By linearity, it suffices to check for $|\psi\rangle = |0\rangle$ and $|\psi\rangle = |1\rangle$.
We back up and set $|\psi\rangle = |1\rangle$ here.



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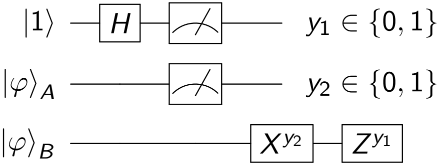

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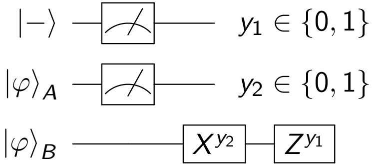

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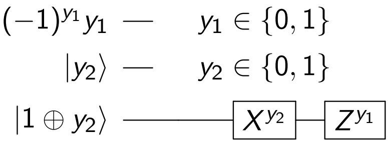

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$$|y_2\rangle \text{ — } y_2 \in \{0,1\}$$

$$(-1)^{y_1} |1\rangle$$

The global phases on the top and bottom wires cancel out. The circuit outputs $|1\rangle$ on the bottom wire, as desired.

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$$\mathrm{tr}_{[n]-A}(|🐱\rangle\langle🐱|) = \frac{1}{2}\left(|0\ldots0\rangle\langle0\ldots0| + |1\ldots1\rangle\langle1\ldots1|\right)$$

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Notice that

$$\mathrm{tr}_{[n]-A}(|🐱^{\perp}\rangle\langle🐱^{\perp}|) = \frac{1}{2}\left(|0\ldots0\rangle\langle0\ldots0| + |1\ldots1\rangle\langle1\ldots1|\right)$$

where

$$|🐱^{\perp}\rangle = \frac{|0\rangle^{\otimes n} - |1\rangle^{\otimes n}}{\sqrt{2}}.$$

Then for any local Hamiltonian H_i acting on A, if

$$\langle🐱| H_i |🐱\rangle = \langle🐱^{\perp}| H_i |🐱^{\perp}\rangle .$$

So, if $|🐱\rangle$ is a ground state, then so is $|🐱^{\perp}\rangle$. Contradiction!

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Let $L \in \text{QMA} = \text{QMA}_{(\frac{2}{3},\frac{1}{3})}$. Idea for amplification, ask for $k = \text{poly}(n)$ copies of the witness state, run verification circuit C_L on each copy, then take classical majority function. Prove two parts: Completeness and Soundness.

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Majority function PSD operators:

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Since, Q_1 is sum of tensor product of P_0 and P_1, Q has an eigenbasis consisting of tensor products of eigenvectors of P_0 and P_1. Therefore, optimal eigenvector for Q_1 is $|\varphi\rangle^{\otimes k}$ where $|\varphi\rangle$ is optima eigenvector for P_1. Classical argument: soundness $\exp(-k)$.