TA session — Day 1

1 Basics: states, measurements, reduced density matrices, trace distance

Exercise 1 (Quantum states and measurements)**.** Consider the EPR state of two qubits,

$$\frac{1}{\sqrt{2}}(|0\rangle \otimes |1\rangle - |1\rangle \otimes |0\rangle)$$

.

- Suppose that we measure the left qubit in the EPR state in an orthononal basis, $|\alpha\rangle, |\alpha^{\perp}\rangle$. What is the probability for the qubit to be projected on $|\alpha\rangle$? Suppose indeed the result is $|\alpha\rangle$, What is the state of the right qubit after the measurement?
- Describe the above measurement using a Hermitian operator M with eigenvalues $+1$ and -1, and compute the expectation value of this measurement.
- What is the reduced density matrix ρ_1 of the left qubit? describe the expectation value of the measurement with respect to the operator M which you defined, using the expression

$$\mathrm{tr}(\rho_1 M).$$

Exercise 2 (entanglement and inner product estimation)**.** Consider the entangled state

$$\frac{1}{\sqrt{2}}(|0\rangle \otimes |\alpha_0\rangle + |1\rangle \otimes |\alpha_1\rangle)$$

The right register may contain many qubits, and the states $|\alpha_0\rangle, |\alpha_1\rangle$ are normalized to 1.

- Calculate the reduced density matrix of the left qubit, as a function of the inner product $\langle\alpha_0|\alpha_1\rangle$.
- Suppose we measure the left qubit with respect to the measurement operator X (whose eigenstates are $|+\rangle, |-\rangle$.) What is the probability to get $|+\rangle$, as a function of $\langle\alpha_0|\alpha_1\rangle$?

Exercise 3. Show that

$$\mathrm{tr}((M_A \otimes I_B)\rho_{AB}) = \mathrm{tr}(M_A \rho_A)$$

where M_A is some (Hermitian) operator on the Hilbert space A, I_B is the identity on the Hilbert space B, ρ_{AB} is some density matrix on the Hilbert space $A \otimes B$ and ρ_A is the reduced matrix of ρ_{AB} to the space A.

Exercise 4 (Distinguishing quantum states)**.** Let σ_0 and σ_1 be two density matrices on the same space. For any projective measurement $\{M_0, M_1\}$ on the same space (i.e. M_0, M_1 are projections such that $M_0 + M_1 = \mathrm{Id}$), we can define the success probability with which M distinguishes σ_0 from σ_1 as

$$p_s(M|\sigma_0, \sigma_1) = \frac{1}{2}\mathrm{Tr}(M_0\sigma_0) + \frac{1}{2}\mathrm{Tr}(M_1\sigma_1) \; ,$$

where the coefficients $\frac{1}{2}$ represent the a priori probability that the state measured is σ_0 or σ_1.

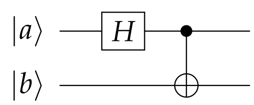
1. Derive a formula for the choice of $\{M_0, M_1\}$ that maximizes $p_s(M|\sigma_0, \sigma_1)$, as a function of the matrices σ_0 and σ_1.

2. Deduce an expression for $\sup_M p_s(M|\sigma_0, \sigma_1)$ as a function of the norm $\|\sigma_0 - \sigma_1\|_1$, where in general $\|X\|_1$ is the sum of the singular values of X. This quantity is called the trace distance, and is a natural measure of distance between density matrices.

3. Compute the trace distance in case $\sigma_0 = |u_0\rangle\langle u_0|$ and $\sigma_1 = |u_1\rangle\langle u_1|$ are pure states. Compare to the Euclidean norm $\||u_1\rangle - |u_2\rangle\|$.

2 Circuits

Exercise 5. Consider the following two-qubit circuit. First, a Hadamard gate is applied on the first qubit. Second, a CNOT gate is applied from the first qubit (the control qubit) to the second qubit.



Write down the four states $|\phi_{ab}\rangle$, for $a, b \in \{0, 1\}$, that are produced by this circuit when the input is $|a\rangle|b\rangle$. Now, choose a single-qubit basis of your choice. Using that basis, compute the following distributions:

1. What is the distribution of outcomes that you obtain by measuring the first output qubit of the circuit only (again, for any input $|a\rangle|b\rangle$)?

2. What distribution do you get by measuring the second output qubit only, in the same basis?

3. What is the joint distribution on outcomes obtained by measuring both output qubits, one at a time, in the same basis?

4. And what if both qubits are measured at the same time?

Exercise 6 (No cloning)**.** Show that there does not exist a 2-qubit circuit that maps any input input state of the form $|\psi\rangle \otimes |0\rangle$, where $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ is an arbitrary single-qubit state, to the state $|\psi\rangle|\psi\rangle$.

Exercise 7. Show that the class BQP is included in PSPACE. *[Hint: express the probability of a cicruit outputting* 1 *as a specific entry in a product of matrices. Use that many linear algebra computations have space-efficient implementation.]* Bonus: improve the inclusion to BQP $\subseteq$ PP. (PP is the class of languages that can be decided by a randomized procedure that provides the correct answer with probability $> \frac{1}{2}$.)

3 Basic quantum algorithms

Exercise 8 (Phase kickback trick)**.** Let f be a function $f : \{0,1\}^n \to \{0,1\}$ and U_f be the unitary oracle which computes f reversibly:

$$U_f|x\rangle|y\rangle = |x\rangle|y \oplus f(x)\rangle \qquad x \in \{0,1\}^n, y \in \{0,1\} \tag{1}$$

Show that one query to U_f suffices to implement the n-qubit unitary V_f defined by

$$V_f|x\rangle = (-1)^{f(x)}|x\rangle \qquad x \in \{0,1\}^n. \tag{2}$$

Exercise 9. The goal of this exercise is to analyze the Deutsch-Josza algorithm in the general case (in class we did a special case where the function outputs only a single bit). In this problem we are given oracle access to a unitary U_f which reversibly computes a function $f : \{0,1\}^n \to \{0,1\}$ as in Eq. (1). We are promised that one of the following two cases holds:

1. The function is constant, i.e., $f(z) = f(y)$ for all $z, y \in \{0,1\}^n$.
2. The function is balanced, i.e., $|\{z : f(z) = 1\}| = |\{z : f(z) = 0\}|$.

The output state of the Deutsch-Josza algorithm is

$$|\psi\rangle = H^{\otimes n} V_f H^{\otimes n} |0^n\rangle.$$

where V_f is defined by Eq. (2). Show that a measurement of all qubits of $|\psi\rangle$ in the computational basis returns the all zeroes string 0^n if and only if case 1 holds (i.e., f is constant).