TA session — Day 1 Selected Solutions

1 Basics: states, measurements, reduced density matrices, trace distance

Exercise 1 (Quantum states and measurements)**.** Consider the EPR state of two qubits,

$$\frac{1}{\sqrt{2}}(|0\rangle \otimes |1\rangle - |1\rangle \otimes |0\rangle)$$

.

- Suppose that we measure the left qubit in the EPR state in an orthononal basis, $|\alpha\rangle, |\alpha^{\perp}\rangle$. What is the probability for the qubit to be projected on $|\alpha\rangle$? Suppose indeed the result is $|\alpha\rangle$, What is the state of the right qubit after the measurement?
- Describe the above measurement using a Hermitian operator M with eigenvalues $+1$ and -1, and compute the expectation value of this measurement.
- What is the reduced density matrix ρ_1 of the left qubit? describe the expectation value of the measurement with respect to the operator M which you defined, using the expression

$$\mathrm{tr}(\rho_1 M).$$

Solution 1. The post-measurement state is given by projecting onto the $|\alpha\rangle$ vector in the left qubit. To perform this projection, we apply the projection operator in the left qubit and the identity operator in the right qubit. The projection operator $|\alpha\rangle\langle\alpha|$ operates like this:

$$|\alpha\rangle\langle\alpha| \, |\psi\rangle = \langle\alpha|\psi\rangle \, |\alpha\rangle \,. \tag{1}$$

$$\begin{aligned} (|\alpha\rangle\langle\alpha| \otimes I)\frac{1}{\sqrt{2}}(|0\rangle \otimes |1\rangle - |1\rangle \otimes |0\rangle) &= \frac{1}{\sqrt{2}}(\langle\alpha|0\rangle \, |\alpha\rangle \otimes |1\rangle - \langle\alpha|1\rangle \, |\alpha\rangle \otimes |0\rangle) & (2)\\ &= \frac{1}{\sqrt{2}} |\alpha\rangle \otimes (\langle\alpha|0\rangle \, |1\rangle - \langle\alpha|1\rangle \, |0\rangle)\,. & (3) \end{aligned}$$

Notice that this projection gave us an unnormalized state. Let's compute its norm, which is the probability for the qubit to be projected onto $|\alpha\rangle$. If we let $|\alpha\rangle = a\,|0\rangle + b\,|1\rangle$, then we can write the state of the right qubit as $a^*\,|1\rangle - b^*\,|0\rangle$. Then you can check that $\| a^*\,|1\rangle - b^*\,|0\rangle \|^2 = a^2 + b^2 = 1$. It follows that the norm-squared of the state in the above equation is $\frac{1}{2}$.

The left state is projected onto $|\alpha\rangle$ with probability $\frac{1}{2}$. The post-measurement state of the right qubit is $a^*\,|1\rangle - b^*\,|0\rangle$.

We say that a ± 1-eigenvalue operator *describes* a two-outcome measurement if the two eigenspaces correspond to the two projections of the measurement. **Here** $M = |\alpha\rangle\langle\alpha| - |\alpha^\perp\rangle\langle\alpha^\perp|$**.**

To compute the reduced density matrix of the left qubit, we perform the partial trace on the right qubit. We'll denote the density matrix of the whole state by ρ, and we'll compute ρ as a sum of simple tensors.

$$\rho = \frac{1}{\sqrt{2}} \left(|0\rangle \otimes |1\rangle - |1\rangle \otimes |0\rangle \right) \frac{1}{\sqrt{2}} \left(\langle 0| \otimes \langle 1| - \langle 1| \otimes \langle 0| \right) \tag{4}$$

$$= \frac{1}{2}(|0\rangle \otimes |1\rangle)(\langle 0| \otimes \langle 1|) - (|0\rangle \otimes |1\rangle)(\langle 1| \otimes \langle 0|) - (|1\rangle \otimes |0\rangle)(\langle 0| \otimes \langle 1|) + (|1\rangle \otimes |0\rangle)(\langle 1| \otimes \langle 0|) \tag{5}$$

$$= \frac{1}{2}(|0\rangle\langle 0| \otimes |1\rangle\langle 1| - |0\rangle\langle 1| \otimes |1\rangle\langle 0| - |1\rangle\langle 0| \otimes |0\rangle\langle 1| + |1\rangle\langle 1| \otimes |0\rangle\langle 0|). \tag{6}$$

Now we wish to apply the partial trace term-by-term. Notice that

$$\mathrm{Tr}\, |0\rangle\langle 0| = \mathrm{Tr}\, |1\rangle\langle 1| = 1, \quad \text{and} \quad \mathrm{Tr}\, |1\rangle\langle 0| = \mathrm{Tr}\, |0\rangle\langle 1| = 0. \tag{7}$$

Then we can compute the reduced density as

$$\rho_1 = \mathrm{Tr}_2\rho = \frac{1}{2}(|0\rangle\langle 0| \cdot (1) - |0\rangle\langle 1| \cdot (0) - |1\rangle\langle 0| \cdot (0) + |1\rangle\langle 1| \cdot (1)) \tag{8}$$

$$= \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|) \tag{9}$$

$$= \frac{1}{2}I. \tag{10}$$

So we can compute

$$\mathrm{Tr}(\rho_1 M) = \frac{1}{2}\mathrm{Tr} M = 0. \tag{11}$$

Exercise 2 (entanglement and inner product estimation)**.** Consider the entangled state

$$\frac{1}{\sqrt{2}}(|0\rangle \otimes |\alpha_0\rangle + |1\rangle \otimes |\alpha_1\rangle)$$

The right register may contain many qubits, and the states $|\alpha_0\rangle, |\alpha_1\rangle$ are normalized to 1.

- Calculate the reduced density matrix of the left qubit, as a function of the inner product $\langle\alpha_0|\alpha_1\rangle$.
- Suppose we measure the left qubit with respect to the measurement operator X (whose eigenstates are $|+\rangle, |-\rangle$.) What is the probability to get $|+\rangle$, as a function of $\langle\alpha_0|\alpha_1\rangle$?

Solution 2. The calculation here is similar to our last calculation of a partial trace, except now we have

$$\mathrm{Tr}\, |\alpha_0\rangle\langle\alpha_1| = \langle\alpha_1|\alpha_0\rangle = \overline{\langle\alpha_0|\alpha_1\rangle}, \quad \mathrm{Tr}\, |\alpha_1\rangle\langle\alpha_0| = \langle\alpha_0|\alpha_1\rangle\,. \tag{12}$$

Just as before we can compute ρ as a density matrix which is a linear combination of simple tensors.

$$\rho = \frac{1}{2}(|0\rangle\langle 0| \otimes |\alpha_0\rangle\langle\alpha_0| + |0\rangle\langle 1| \otimes |\alpha_0\rangle\langle\alpha_1| + |1\rangle\langle 0| \otimes |\alpha_1\rangle\langle\alpha_0| + |1\rangle\langle 1| \otimes |\alpha_1\rangle\langle\alpha_1|). \tag{13}$$

Taking the partial trace gives us

$$\mathrm{Tr}_B\rho = \frac{1}{2}(|0\rangle\langle 0| + \overline{\langle\alpha_0|\alpha_1\rangle}\,|0\rangle\langle 1| + \langle\alpha_0|\alpha_1\rangle\,|1\rangle\langle 0| + |1\rangle\langle 1|). \tag{14}$$

We can write this as a two-by-two matrix:

$$\mathrm{Tr}_B\rho = \frac{1}{2}\begin{pmatrix} 1 & \overline{\langle\alpha_0|\alpha_1\rangle} \\ c & \langle\alpha_0|\alpha_1\rangle \end{pmatrix} 1 \tag{15}$$

Exercise 3. Show that

$$\mathrm{tr}((M_A \otimes I_B)\rho_{AB}) = \mathrm{tr}(M_A\rho_A)$$

where M_A is some (Hermitian) operator on the Hilbert space A, I_B is the identity on the Hilbert space B, ρ_{AB} is some density matrix on the Hilbert space $A \otimes B$ and ρ_A is the reduced matrix of ρ_{AB} to the space A.

Solution 3. Recall that $\mathrm{tr}(A \otimes B) = \mathrm{tr}(A) \cdot \mathrm{tr}(B)$. This can be easily seen by noticing that the elements on the diagonal of $A \otimes B$ are the pair-wise product of the element on the diagonal of A and diagonal of B. Consider a generalized state ρ_{AB}. It can be expressed as

$$\rho_{AB} = \sum_i p_i\rho_i \otimes \sigma_i$$

where $\{p_i\}$ here is a probability distribution, ρ_i acts on A, and σ_i acts on B. Then,

$$\rho_A = \sum_i p_i\rho_i \cdot \mathrm{tr}(\sigma_i).$$

Then, the left hand side of our target equality can be expressed as

$$\begin{aligned} \mathrm{tr}((M_A \otimes I_B)\rho_{AB}) &= \mathrm{tr}\left((M_A \otimes I_B)\sum_i p_i\rho_i \otimes \sigma_i\right) \\ &= \sum_i p_i \mathrm{tr}(M_A\rho_i \otimes \sigma_i) \\ &= \sum_i p_i \mathrm{tr}(M_A\rho_i) \cdot \mathrm{tr}(\sigma_i). \end{aligned}$$

The first line is application of the definition, the second the linearity of the trace operator, and the third the trace rule over tensor products previously mentioned.

The right hand side of our target equality can be expessed as

$$\begin{aligned} \mathrm{tr}_B(M_A\rho_A) &= \mathrm{tr}_B\left(M_A \sum_i p_i\rho_i \cdot \mathrm{tr}(\sigma_i)\right) \\ &= \sum_i p_i \mathrm{tr}(\sigma_i) \cdot \mathrm{tr}(M_A\rho_i). \end{aligned}$$

Clearly, the sides are equal.

Equivalently, we did not have to express the density matrix ρ_{AB} in such verboseness. We could have also relied on the linearity[1] *of the partial trace operator* tr_B. *In which case we could have proved the statement for $\rho_{AB} = \rho_A \otimes \rho_B$ and extended it by linearity.*

[1]We leave proving the linearity as an exercise. It is similar to the proof for generic trace operator.