LOG-RANK & LIFTING FOR AND-FUNCTIONS

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LOG-RANK CONJECTURE

$F : \{0,1\}^n \times \{0,1\}^n \to \{0,1\}$

Alice

$x \in \{0,1\}^n$

protocol $\Pi(x,y)$

$m_1 \in \{0,1\}$ →

← $m_2 \in \{0,1\}$

⋮

$F(x,y)$

Bob

$y \in \{0,1\}^n$

$CC(F)$ = length of shortest Π computing F

Claim: $\log \operatorname{rank}(F) \le CC(F)$

Log-rank conjecture:

$$CC(F) \le (\log \operatorname{rank}(F))^{O(1)}$$

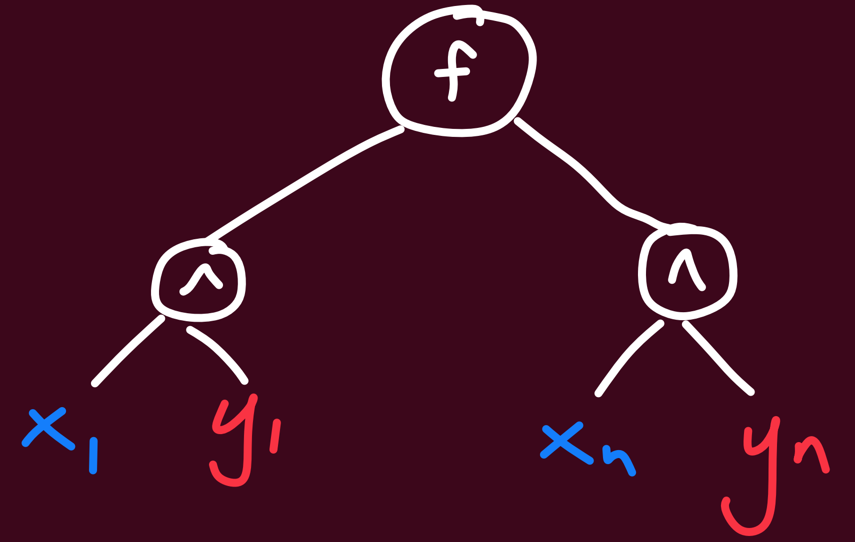
Wide open!

Lovett: $CC(F) \lesssim \sqrt{\operatorname{rank}(F)}$

AND-FUNCTIONS

$f : \{0,1\}^n \to \{0,1\}$

$f_\wedge(x,y) = f(x \wedge y)$



Claim: $CC(f_\wedge) \le 2\, DT_\wedge(f)$

$DT_\wedge(f) \approx$ "AND-decision tree complexity"

Claim: $\operatorname{rank}(f_\wedge) = \operatorname{spar}(f)$

$\operatorname{spar}(f) \approx$ "# of non-zero monomials"

Main results: Any $f : \{0,1\}^n \to \{0,1\}$

1. $DT_\wedge(f) \le CC(f_\wedge)^{O(1)} \cdot \log n$
2. $CC(f_\wedge) \ll (\log \operatorname{rank}(f_\wedge))^{O(1)} \cdot \log n$

Most general known case!

STRUCTURE OF BOOLEAN FUNCTIONS

$f(x) = \sum_{S \subseteq [n]} \alpha_S x_S, \quad x_S = \prod_{i \in S} x_i$

$\operatorname{spar}(f) = \left| \{ S \subseteq [n] \mid \alpha_S \neq 0 \} \right|$

Main technical lemma: For any $f : \{0,1\}^n \to \{0,1\}$, $\operatorname{spar}(f) = r$, there is a hitting set w/ size $(\log r)^{O(1)}$ (no dependence on n!)

Proof uses Nisan-Szegedy, LP duality, randomized rounding argument

Lemma: $\operatorname{spar}(f) = r$ + set of size $(\log r)^{O(1)}$ hitting each monomial

$\implies DT_\wedge(f) = (\log r)^{O(1)} \cdot \log n$

Main open problem: remove $\log n$!