Russell's course, lec 2.

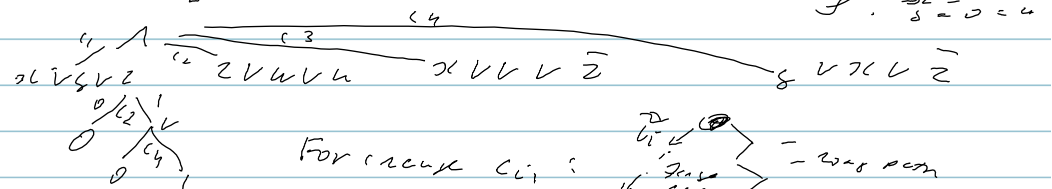
Switching lemma:

Let F be a k-CNF fmla. Pick rand rest ρ leaving pn vars unset.

$\exists\, c>0$ s.t. $P[\,CDT[F|_\rho] \ge D\,] \le (cpk)^D$

Proof idea: we'll code each ρ with $CDT[F|_\rho] \ge D$ by a rest. ρ_{code} leaving $pn-D$ vars unset and a hint, a short message of length $D\,(2\log k + 2)$ bits. There are relatively few $(\rho_{code}, \text{hint})$ pairs possible, so few ρs have $CDT[F|_\rho] \ge D$

ρ: $x=1$, $s=0=u$



For clause c_{i_1}:

h_1: $(3,0,1)$ $(5,1,1)$, $(7,0,0)$.

x set to 0 in more than i'th clause. $(i, l_1, l_2) \in [k]$ — $2\log k + 2$ bits

On b_1, long path in clause c_{i_1}, then same for c_{i_2}..

h_2: same for second unset clause c_{i_2}. Visit $D' \le D$ clauses.

$\rho_{code} = \rho \circ t_1 \circ \dots \circ t_{d'}$ has $pn-D$ vars unset. $|hint| = D(\log k + 2)$

decoding: look at $F|_{\rho_{code}}$. Let c_{i_1} be first clause falsified. Look at hint to see what var in c_{i_1} were unset, and how they were set in b_1. Temporarily set them to values in b_1. Repeat until D vars unset have been recovered

ρ_{random}: ρ set D vars in random, h_r: random sequence of same size, $n-pn+D$ bits, $n-D$ of them are "impostors". Prob. of guessing them: $\frac{D}{n-pn+D}\cdot\frac{D-1}{n-pn+D-1} \le \left(\frac{D}{n-pn}\right)^D$

Given ρ, there are $\binom{pn}{D}\cdot 2^D$ ρ_{random}; prob of ρ_{rand} correct $1/$this

D prob. that hint correct: $(1/4k)^D$

$\binom{D}{p_n}$ $\text{prob}[\,\text{decode}[\rho_{rand}, h_{rand}] = \rho\,] \ge \text{Prob}[\,CDT[F|_\rho] \ge D\,]\cdot \text{Prob}[\rho_r = \rho_c]$

$\cdot\ \text{prob}[h_r = h] = \text{Prob}[CDT[F|_\rho] \ge D]\cdot 1/\binom{pn}{D}\cdot 2^D\cdot(4k)^D \le$

$\le \left(\frac{epn}{D}\right)^D \cdot 2^D\cdot(4k)^d \cdot D^D/(n-pn)^D = (8epk/(1-p))^D = (cpk)^D$

Applications of switching lemma:

1) lb on AC^0 circs for parity

Lemma: let $\mathcal{C}$ be depth d circ s.t. # gates at top $d-1$ levels $\leq S$, and fan-in at bottom $\leq \log S$

If $\mathcal{C}$ computes parity of n vars, $S \geq 2^{\Omega(n^{1/d-1})}$

Proof ind on d. Base: $d=2$: last class exercise

Ind. step: Restr. $\mathcal{C}$ by ρ that leaves small const $c'n/\log S$ vars unset. $\mathcal{C}$ has $\leq S$ CNF/DNF that are $\log S$-CNF. Prob. that any of these $\mathcal{C}_i$ has $P[CDT[\mathcal{C}_i|\rho] \geq \log S] \leq \left(\frac{c \cdot c'}{\log S} \cdot \log S\right)^{\log S} < \frac{1}{S}$

So $\exists \rho$ s.t. $\forall \mathcal{C}_i$, $CDT[\mathcal{C}_i|\rho] \leq \log S$

So replacing each CDT w/ a DNF and merging at next higher level set $\mathcal{C}|_\rho$ has $\leq S$ gates at levels $d-1 \ldots 1$, and $\log S$ fan-in at bottom

This should compute parity of ρn vars. Apply induction w/ $d-1$ and $n' = cn/\log S$. So $n \leq (\log S)^{d-1}$, $n' \leq c(\log S)^{d-2}$

$n \leq \log(\log S)^{d-1}$

2) k-CNF counting: count # of x s.t. $\mathcal{C}(x)=1$, given k-CNF $\mathcal{C}$.

Alg: partition vars into groups $|X| = (1-c'/k)n$, $|Y| = \frac{c'n}{k}$

For each setting of vars in X, look at corresp. rest. ρ.

Create $CDT[\mathcal{C}|_\rho]$ and count # of sols.

Exp time of the alg: $\sum_{\substack{\text{sets } \rho \\ \text{to } X}} O(|CDT(\mathcal{C}|_\rho)|) \approx 2^{(n-pn)}\left(E_{\rho}\, CDT[\mathcal{C}|_\rho]\right)$

$\sum 2^{\ell}\, \text{Prob}[CDT(\mathcal{C}|_\rho) \geq \ell] \leq O(2^{n-pn})$

where $p \approx \frac{c'}{k}$

Next class: learning from rand samples.