Enumerating split-pair arrangements

Ron Graham*† Nan Zang*

June 17, 2007

Abstract

An arrangement of the multi-set $\{1, 1, 2, 2, \ldots, n, n\}$ is said to be "split-pair" if for all $i < n$, between the two occurrences of i there is exactly one $i + 1$. We enumerate the number of split-pair arrangements and in particular show that the number of such arrangements is $(-1)^{n+1}2^n(2^{2n} - 1)B_{2n}$ where B_i is the ith Bernoulli number.

1 Introduction

In robotic scheduling, a single robot arm moves identical product along through a predetermined sequence of machines labelled $M_1, M_2, \ldots, M_n, M_{n+1}$. The first machine M_1 is an unlimited supply of the raw product, the last machine M_{n+1} is unlimited storage for the finished product, while the remaining machines can only support one product at a time. In going from the raw product to the finished product, each machine must be visited in turn.

Associated to each machine there is a time for processing the product and in addition there is a time cost in moving the robotic arm between machines. A natural question to ask is what is the most efficient cyclic sequence of moves for the robotic arm to make to maximize the *average throughput* of the products. By a move we mean moving a product from machine M_i (after the product has been processed on that machine) to machine M_{i+1} (which is currently empty), such a move will be denoted by a_i.

For example, one possible sequence of moves is for the arm to move in the sequence $a_1a_2 \ldots a_n$, i.e., it takes a product from start to finish waiting at each machine for

*{rgraham, nzang}@cs.ucsd.edu, Department of Computer Science, University of California at San Diego, La Jolla, CA 92093-0404

†Research supported in part by NSF Grant CCR-0310991

it to be processed before moving to the next one. However, instead of waiting for a product to process at a machine the robotic arm is allowed to move to other machines which may have product ready to move to the next machine and thus increase the average throughput.

The assumption of being cyclic means that each a_i, $1 \leq i \leq n$, occurs equally often in the cycle (since otherwise we would have a "collision" of products on some machine). An r-unit cycle is one for which the number of times that each action occurs during the cycle is r. (For a comprehensive survey of robotic scheduling, the reader is referred to [1].)

It is easy to see that the number of possible 1-unit cycles is $n!$ (i.e., any permutation of the actions gives a valid 1-cycle). It is not quite so easy to see how many valid 2-unit cycles there are. This is because not every arrangement of the actions $\{a_1, a_1, a_2, a_2, \ldots, a_n, a_n\}$ gives a valid cycle. The problem arises in that in order for action a_i to be completed, the machine M_{i+1} must currently be empty or $i = n$. What this translates into is that between the two occurrences of a_i in the arrangement of the action there must be exactly one occurrence of the action a_{i+1} (i.e., to empty machine M_{i+1}) or $i = n$. This leads naturally to *split-pair* arrangements. A split-pair arrangement of the multi-set $D_n := \{1, 1, 2, 2, 3, 3, \ldots, n, n\}$ is an arrangement $x_1 x_2 x_3 \ldots x_{2n}$ of the elements of D_n such that for each $i < n$, the two occurrences of i are separated by exactly one occurrence of $i + 1$.

We denote by s_n the number of split-pair arrangements of D_n. In Table 1 we have listed all the split-pair arrangements for $n = 1, 2, 3$ and thus we have $s_1 = 1$, $s_2 = 2$, and $s_3 = 12$.

$n = 1$	11
$n = 2$	1212, 2121
$n = 3$	121323, 123123, 132132, 132312, 213213, 213231, 231213, 231231, 312132, 312312, 321321, 323121

Table 1: Split-pair arrangements for D_1, D_2 and D_3.

Our main result is to derive a simple expression for s_n, namely we have the following.

Theorem 1. *For $n \geq 1$, $s_n = (-1)^{n+1} 2^n (2^{2n} - 1) B_{2n}$ where B_i is the ith Bernoulli number.*

More information about Bernoulli numbers can be found in the appendix as well as in many introductory texts (i.e., see [4]).

2 Prelimaries

Our proof of Theorem 1 will follow the proof of a related problem given in [3]. Namely, we derive recurrences for the quantities of interest, define a generating function, show that this generating function satisfies a certain partial differential equation (PDE), and then identify specific coefficients arising from this PDE to get the final result. Before we begin we will find it useful to refine our count by counting split-pair arrangements according to the location(s) of the n, as well as introduce some matrices.

Let $s_n(i)$, denote the number of split-pair arrangements $x_1x_2x_3\ldots x_{2n}$ of D_n for which $x_i = n$. Similarly, let $s_n(i,j)$ denote the number of split-pair arrangements of D_n for which $x_i = x_j = n$. Note that if $x_1x_2x_3\ldots x_{2n}$ is a split-pair arrangement then so is the cyclic permutation $x_2x_3\ldots x_{2n}x_1$ (and by induction, this is true for any cyclic permutation $x_kx_{k+1}\ldots x_{k-1}$). In addition, the reverse arrangement $x_{2n}x_{2n-1}\ldots x_1$ is also a split-pair arrangement. These observations imply the following results.

Lemma 2. *For any i, $s_n = ns_n(i)$.*

Lemma 3. *If $|i-j| = |i'-j'|$ or $|i-j| = 2n - |i'-j'|$ then $s_n(i,j) = s_n(i',j')$.*

Let S_n denote the matrix with (i,j) entry equal to $s_n(i,j)$. Several examples are given below.

$$S_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad S_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \qquad S_3 = \begin{pmatrix} 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 \\ 1 & 0 & 0 & 0 & 1 & 2 \\ 2 & 1 & 0 & 0 & 0 & 1 \\ 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \end{pmatrix},$$

$$S_4 = \begin{pmatrix} 0 & 0 & 4 & 8 & 10 & 8 & 4 & 0 \\ 0 & 0 & 0 & 4 & 8 & 10 & 8 & 4 \\ 4 & 0 & 0 & 0 & 4 & 8 & 10 & 8 \\ 8 & 4 & 0 & 0 & 0 & 4 & 8 & 10 \\ 10 & 8 & 4 & 0 & 0 & 0 & 4 & 8 \\ 8 & 10 & 8 & 4 & 0 & 0 & 0 & 4 \\ 4 & 8 & 10 & 8 & 4 & 0 & 0 & 0 \\ 0 & 4 & 8 & 10 & 8 & 4 & 0 & 0 \end{pmatrix}$$

The structure of these matrices is a consequence of the symmetry of $s_n(i,j)$ together with the result of Lemma 3. Note in particular that the rows are cyclic shifts of each other, so it suffices to know only the first row of S_n. We thus collect all the first rows of the S_n to form the (infinite) matrix A defined entrywise by

$$A(n,k) = s_n(1,k+1),$$

for $1 \le k < 2n$, and 0 otherwise. A portion of A is shown below, where the rows are indexed by $n = 1, 2, \ldots$ and the columns are indexed by $k = 1, 2, \ldots$.

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 4 & 8 & 10 & 8 & 4 & 0 & 0 & 0 & \ldots \\ 0 & 34 & 68 & 94 & 104 & 94 & 68 & 34 & 0 & \ldots \\ 0 & 496 & 992 & 1420 & 1712 & 1816 & 1712 & 1420 & 992 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{pmatrix}$$

The matrix A will play a central role in the proof of Theorem 1. So we will now focus on establishing properties of A, most important among them a recurrence for the entries of A (see Theorem 4).

First, since by Lemma 3, $s_n(1, k+1) = s_n(1, 2n-k+1)$, we see that $A(n, k) = A(n, 2n-k)$ for $1 \le k \le 2n-1$.

We next claim that $A(n, 3) = 2A(n, 2)$ for $n \ge 3$. To see this, suppose that $nxny \ldots wz$ is a split-pair arrangement for D_n which is counted by $s_n(1, 3)$. Then we must have $x = n-1$ and $y, z \ne n-1$. Hence, we can form *two* split-pair arrangements counted by $s_n(1, 4)$, namely, $nz(n-1)ny \ldots w$ and $n(n-1)yn \ldots wz$. Furthermore, any split-pair arrangement counted by $s_n(1, 4)$ must arise this way. Consequently,

$$A(n, 3) = s_n(1, 4) = 2s_n(1, 3) = 2A(n, 2), \quad n \ge 3. \tag{1}$$

Next note that if $(n-1)x \ldots$ is a split-pair arrangement for D_{n-1} then we can place two n's on each side of the initial $n-1$ to form $n(n-1)nx \ldots$, which is a split-pair arrangement for D_n which is counted by $s_n(1, 3)$ (and conversely, any such split-pair arrangement can be generated this way). This shows:

$$A(n, 2) = s_n(1, 3) = \sum_{k \ge 1} s_{n-1}(1, k) = \sum_{k \ge 1} A(n-1, k), \quad n \ge 2. \tag{2}$$

Finally, it is easy to see from the definitions of $A(n, k)$ that

$$A(1, 1) = 1, \quad A(k, 1) = 0 = A(1, k), \quad k \ge 2. \tag{3}$$

Observe that if $x_1 x_2 \ldots x_{2n}$ is a split-pair arrangement for D_n then by removing the two occurrences of n, we are left with a split-pair arrangement for D_{n-1}. We are interested in going the other way. The question is this: for a given split-pair arrangement $X = x_1 x_2 \ldots x_{2n-2}$ for D_{n-1}, in how many ways can we *insert* two n's in order to form a split-pair arrangement for D_n? Of course, this will depend on the location of the $(n-1)$'s in X.

We will count the number of ways we can form a split-pair arrangement for D_n from X which has its first entry equal to n (and so, is counted by $s_n(1)$). So suppose we insert n's as follows into X to form $X' = x'_1 x'_2 \dots x'_{2n} = n x_1 x_2 \dots x_{j-2} n x_{j-1} \dots x_{2n-2}$. Thus, $x'_1 = x'_j = n$. If the indices in X' where the two $(n-1)$'s occur are u and v with $u < v$, so that $x'_u = x'_v = n-1$ then in order for X' to a split-pair arrangement for D_n it is necessary and sufficient that

$$u + 1 < j < v + 2$$

or equivalently,

$$u \le j - 2, \quad v \ge j - 1.$$

From the above, it follows that

$$s_n(1, j) = \sum_{\substack{u \le j-2 \\ v \ge j-1}} s_{n-1}(u, v) = \sum_{\substack{1 \le u \le j-2 \\ 1 \le v \le 2n-2}} s_{n-1}(u, v) - \sum_{\substack{1 \le u \le j-2 \\ 1 \le v \le j-2}} s_{n-1}(u, v).$$

If we now define $t_m(j)$ by

$$t_m(j) = \sum_{k=1}^{j} s_m(1, k)$$

then it follows from the structure of S_{n-1} that

$$\sum_{\substack{1 \le u \le j-2 \\ 1 \le v \le j-2}} s_{n-1}(u, v) = 2 \sum_{k=1}^{j-2} t_{n-1}(k).$$

Consequently we have

$$s_n(1, j) = (j - 2)\, t_{n-1}(2n - 2) - 2 \sum_{k=1}^{j-2} t_{n-1}(k), \quad j \ge 2. \tag{4}$$

Theorem 4. *We have*

$$A(n, k) = 2\, A(n, k - 1) - A(n, k - 2) - 2\, A(n - 1, k - 2),$$

for $n \ge 2$*, and* $3 \le k \le 2n + 1$.

Proof. Using (4) we have for $3 \le k \le 2n$,

$$
\begin{aligned}
A(n,k) - 2A(n,k-1) + A(n,k-2) &= s_n(1,k+1) - 2s_n(1,k) + s_n(1,k-1) \\
&= (k-1)\, t_{n-1}(2n-2) - 2\sum_{i=1}^{k-1} t_{n-1}(i) \\
&\quad -2(k-2)\, t_{n-1}(2n-2) + 4\sum_{i=1}^{k-2} t_{n-1}(i) \\
&\quad +(k-3)\, t_{n-1}(2n-2) - 2\sum_{i=1}^{k-3} t_{n-1}(i) \\
&= -2t_{n-1}(k-1) + 2t_{n-1}(k-2) \\
&= -2\sum_{j=1}^{k-1} s_{n-1}(1,j) + 2\sum_{j=1}^{k-2} s_{n-1}(1,j) \\
&= -2s_{n-1}(1,k-1) \;=\; -2A(n-1,k-2),
\end{aligned}
$$

as claimed. For $k = 2n+1$ the result follows from (1) by symmetry. □

The final matrix we will need is the (infinite) matrix A' defined as follows:

$$
A'(n,k) \;=\; \begin{cases} \dfrac{A(n/2+1,k)}{2^{n/2}} & \text{for } n \text{ even, } 1 \le k \le n, \\ 0 & \text{otherwise.} \end{cases} \tag{5}
$$

A part of A' is shown below.

$$
A' = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\
0 & 1/2 & 0 & 0 & 0 & 0 & 0 & \dots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\
0 & 1/4 & 1/2 & 1/4 & 0 & 0 & 0 & \dots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\
0 & 1/2 & 1 & 5/4 & 1 & 1/2 & 0 & \dots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots
\end{bmatrix}
$$

It follows from (1), (2), (3), Lemma 3 and Theorem 4 that A' satisfies the following:

(i) $A'(n,1) = 0$, for $n \ge 1$,

(ii) $A'(n,2) = A'(n,n)$, for $n \ge 2$,

(iii) $A'(n,3) = A'(n,n-1) = 2A'(n,2)$, for $n \geq 3$,

(iv) $A'(n+2,2) = \frac{1}{2}\sum_{k=1}^{n} A'(n,k)$, for $n \geq 2$,

(v) $A'(n,k) = 2A'(n,k-1) - A'(n,k-2) - A'(n-2,k-2)$, for $n \geq 3$ and $3 \leq k \leq n+1$.

3 Proof of Theorem 1

From the definitions above along with Lemma 3 and (2), we note that

$$s_n = ns_n(1) = n\sum_{k\geq 1} s_n(1,k) = nA(n+1,2) = n2^n A'(2n,2).$$

To finish the proof of Theorem 1 it suffices to show that for $n \geq 1$

$$A'(2n,2) = (-1)^{n-1}\frac{2^{2n}-1}{n}B_{2n}. \tag{6}$$

It is this equality which we shall establish.

Proof of Theorem 1. The proof will proceed by a number of steps. First, we define the (mixed) generating function $G(x,y)$ by:

$$G(x,y) = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} A'(n,k)\frac{x^n}{n!}y^k.$$

We now rewrite $G(x,y)$ using the recurrence for $A'(n,k)$:

$$
\begin{aligned}
G(x,y) &= \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} A'(n,k)\frac{x^n y^k}{n!} \;=\; \sum_{n=1}^{\infty}\sum_{k=2}^{n+1} A'(n,k)\frac{x^n y^k}{n!} \\
&= y^2 \sum_{n=1}^{\infty} A'(n,2)\frac{x^n}{n!} + \sum_{n=4}^{\infty}\sum_{k=3}^{n+1} A'(n,k)\frac{x^n y^k}{n!} \\
&= y^2 \sum_{n=1}^{\infty} A'(n,2)\frac{x^n}{n!} \\
&\quad + \sum_{n=4}^{\infty}\sum_{k=3}^{n+1} \big(2A'(n,k-1) - A'(n,k-2) - A'(n-2,k-2)\big)\frac{x^n y^k}{n!} \\
&= y^2 \sum_{n=1}^{\infty} A'(n,2)\frac{x^n}{n!} + 2y \sum_{n=4}^{\infty}\sum_{k=2}^{n} A'(n,k)\frac{x^n y^k}{n!} \\
&\quad - y^2 \sum_{n=4}^{\infty}\sum_{k=1}^{n-1} A'(n,k)\frac{x^n y^k}{n!} - y^2 \int_0^x \int_0^v G(u,y)\,du\,dv \\
&= A'(2,2)\frac{x^2y^2}{2!} + y^2 \sum_{n=4}^{\infty} A'(n,2)\frac{x^n}{n!} \\
&\quad + 2y\Bigg(\sum_{n=1}^{\infty}\sum_{k=1}^{n} A'(n,k)\frac{x^n y^k}{n!} - A'(2,2)\frac{x^2y^2}{2!}\Bigg) \\
&\quad - y^2\Bigg(\sum_{n=1}^{\infty}\sum_{k=1}^{n} A'(n,k)\frac{x^n y^k}{n!} - A'(n,n)\frac{x^n y^n}{n!}\Bigg) \\
&\quad - y^2 \int_0^x \int_0^v G(u,y)\,du\,dv.
\end{aligned}
$$

It now follows (with some computation) that:

$$
\begin{aligned}
G(x,y) &= \frac{1}{4}x^2y^2(1-y)^2 + y^2 \sum_{n=4}^{\infty} A'(n,2)\frac{x^n}{n!} + y^2 \sum_{n=4}^{\infty} A'(n,n)\frac{x^n y^n}{n!} \\
&\quad + (2y - y^2)G(x,y) - y^2 \int_0^x \int_0^v G(u,y)\,du\,dv.
\end{aligned}
$$

Differentiating the above expression for $G(x,y)$ twice with respect to x (and using (ii)), we have the following differential equation for $G(x,y)$:

$$(1-y)^2\frac{\partial^2}{\partial x^2}G(x,y)+y^2G(x,y) = \frac{1}{2}y^2(1-y)^2+y^2\sum_{n=2}^{\infty}A'(n+2,2)\frac{x^n}{n!} +y^4\sum_{n=2}^{\infty}A'(n+2,2)\frac{x^ny^n}{n!}. \quad (7)$$

with initial conditions $G(0,y)=0$ and $\frac{\partial G}{\partial x}(0,y)=0$.

Now, set

$$c_n(y)=\sum_{k=1}^{n}A'(n,k)y^k$$

so that

$$G(x,y)=\sum_{n=1}^{\infty}c_n(y)\frac{x^n}{n!}.$$

By identifying coefficients of powers of x in (7), we obtain the recurrences

$$(1-y)^2c_2(y)=\frac{1}{2}y^2(1-y)^2,$$

$$(1-y)^2c_{n+2}(y)+y^2c_n(y)=y^2A'(n+2,2)+y^{n+4}A'(n+2,2), \quad n\geq 2.$$

It now follows by induction on n that for $n\geq 1$

$$c_{2n}(y) = \frac{1}{2}(-1)^{n-1}\frac{y^{2n}}{(1-y)^{2n-2}}+\sum_{k=2}^{n}(-1)^{n-k}A'(2k,2)\Big(\frac{y}{1-y}\Big)^{2n-2k+2} +y^{2n+2}\sum_{k=2}^{n}(-1)^{n-k}A'(2k,2)\frac{1}{(1-y)^{2n-2k+2}}. \quad (8)$$

Expanding powers of $1-y$ using the binomial theorem, we can rewrite (8) as

$$\begin{aligned} c_{2n}(y) &= \sum_{k=2}^{2n}A'(2n,k)y^k=\frac{1}{2}\sum_{i=0}^{\infty}(-1)^{n-1}y^{2n+i}\binom{2n+i-3}{i} \\ &+\sum_{j=2}^{n}\sum_{i=0}^{\infty}(-1)^{n-j}A'(2j,2)y^{2n-2j+2+i}\binom{2n-2j+i+1}{i} \\ &+\sum_{j=2}^{n}\sum_{i=0}^{\infty}(-1)^{n-j}A'(2j,2)y^{2n+2+i}\binom{2n-2j+i+1}{i}. \end{aligned}$$

Equating coefficients of y^k on the right and left sides of the above now yields

$$A'(2n,k)=\sum_{j=2}^{n}(-1)^{n-j}\binom{k-1}{2n-2j+1}A'(2j,2),\quad \text{for } 2\le k\le 2n-1, \tag{9}$$

and

$$A'(2n,2n)=\frac{1}{2}(-1)^{n-1}+\sum_{j=2}^{n}(-1)^{n-j}\binom{2n-1}{2n-2j+1}A'(2j,2). \tag{10}$$

Finally, we use induction on n to prove that (6) holds. Since $A'(2,2)=\frac{1}{2}=\frac{2^{2*1}-1}{1}(-1)^{1-1}B_2$ then (6) holds for $n=1$. We now assume that (6) holds for $n\le N$, and show it also holds for $n=N+1$:

$$\begin{aligned}
A'(2N+2,2) &= \frac{1}{2}\sum_{k=1}^{2N}A'(2N,k)\\
&= \frac{1}{2}\sum_{k=1}^{2N}\sum_{j=1}^{N}(-1)^{N-j}\binom{k-1}{2N-2j+1}A'(2j,2)\\
&= \frac{1}{2}\sum_{k=1}^{2N}\sum_{j=1}^{N}(-1)^{N-j}\binom{k-1}{2N-2j+1}\frac{2^{2j}-1}{j}(-1)^{j-1}B_{2j}\\
&= (-1)^{N-1}\sum_{j=1}^{N}\binom{2N}{2j-2}\frac{2^{2j}-1}{2j}B_{2j}\\
&= (-1)^{(N+1)-1}\frac{2^{2(N+1)}-1}{(N+1)}B_{2N+2}.
\end{aligned}$$

The second line follows from (9) and (10) with some simplification. Line three uses the induction hypothesis while line four involves a binomial coefficient sum. Line five follows from Lemma 5 found in the Appendix. This completes the induction step and the proof of (6) is complete, which concludes the proof of Theorem 1. □

4 Concluding remarks.

The sequence $(s_n(1):n\ge 1)=(1,1,4,34,496,\ldots)$ has occurred several times previously in the literature (the reader is referred to [7] for a comprehensive collection of integer sequences). For example, it arose in the study of strictly ordered binary trees in [6] and in the enumeration of regular unimodular triangulations in [2]. It also occurs in [5] where it counts the number of so-called 0-1-2 increasing trees on $2n-1$ vertices with n end-vertices. So far, no one has managed to establish explicit bijections between any pair of these classes of objects. It would certainly be of interest to do so.

One might also look at variations of our original problem of enumerating split-pair arrangements of $\{1,1,2,2,\ldots,n,n\}$. For example, one might require in addition that between the two occurrences of n there is exactly one occurrence of 1 (which makes the original problem more symmetric), in terms of robotic scheduling this could correspond to a product having to go through a cycle of machines several times before it is completed. We have not attempted to enumerate such arrangements in this note (the corresponding sequence did not previously appear in [7]).

Another natural generalization coming out from the original robotic scheduling problem (for 3-unit cycles) would be to consider the same enumeration problem but with $D_n = \{1,1,1,2,2,2,\ldots,n,n,n\}$ (or more generally with k copies of each integer). We hope to return to some of these problems at a later time.

Acknowledgements

The authors express their appreciation to Steve Butler, Fan Chung and Alexander Postnikov for useful discussions and comments made in the preparation of this note.

References

[1] M. Dawande, H. N. Geisman, S. Sethi and C. Sriskandarajah, Sequencing and scheduling in robotic cells: recent developments, *Jour. of Scheduling* **8** (2005), 387–426.

[2] M. P. Develin and S. P. Sullivant, Markov Bases of Binary Graph Models, *Annals of Combinatorics* **7** (2003), 441–466.

[3] R. C. Entringer, A combinatorial interpretation of the Euler and Bernoulli numbers, *Nieuw Arch. Wisk. (3)* **14** (1966), 241–246.

[4] **Concrete Mathematics, 2nd edition**, R. L. Graham, D. E. Knuth and O. Patashnik, Reading, Massachusetts: Addison-Wesley, (1994), xiii+657 *pp.*

[5] A. Kuznetsov, I. Pak and A. Postnikov, Increasing trees and alternating permutations, *Uspekhi Mat. Nauk* **49** (1994), 79–110.

[6] C. Poupard, Deux proprietes des arbres binaires ordonnes stricts, *European J. Combin.* **10** (1989), 369–374.

[7] N. J. A. Sloane, **On-line Enyclopedia of Integer Sequences**, `http://www.research.att.com/~njas/sequences/`

Appendix

In this section we present an identity involving Bernoulli numbers which was needed in the preceding arguments (where we include a few extra details for the reader's benefit). Recall that the Bernoulli numbers B_n can be defined by the following exponential generating function:

$$\frac{z}{e^z - 1} = \sum_{n\geq 0} B_n \frac{z^n}{n!}.$$

Thus, we have $B_0 = 1, B_1 = -1/2, B_2 = 1/6$, etc., and in particular, $B_{2k+1} = 0$ for all $k \geq 1$ (for more information, see [4]).

Lemma 5. *For $n \geq 2$, we have*

$$\sum_{j=1}^{n-1} \binom{2n-2}{2j-2} \frac{4^j - 1}{2j} B_{2j} = -\frac{4^n - 1}{n} B_{2n}.$$

Proof.

$$\begin{aligned}
\sum_{n\geq 0} 4^n(4^n - 1)B_{2n}\frac{z^{2n-1}}{(2n)!} &= \frac{1}{z}\sum_{n\geq 0} B_{2n}\frac{(4z)^{2n}}{(2n)!} - \frac{1}{z}\sum_{n\geq 0} B_{2n}\frac{(2z)^{2n}}{(2n)!} \\
&= \frac{1}{z}\Big(\sum_{n\geq 0} B_n \frac{(4z)^n}{(n)!} - 4zB_1\Big) - \frac{1}{z}\Big(\sum_{n\geq 0} B_n\frac{(2z)^n}{(n)!} - 2zB_1\Big) \\
&= \frac{1}{z}\Big(\frac{4z}{e^{4z} - 1} + \frac{1}{2}4z\Big) - \frac{1}{z}\Big(\frac{2z}{e^{2z} - 1} + \frac{1}{2}2z\Big) \\
&= \frac{4}{e^{4z} - 1} - \frac{2}{e^{2z} - 1} + 1 \\
&= \frac{e^{4z} - 2e^{2z} + 1}{e^{4z} - 1} = \frac{e^z - e^{-z}}{e^z + e^{-z}} = \tanh(z).
\end{aligned}$$

Therefore, we have

$$\tanh(z/2) = \frac{e^z - 1}{e^z + 1} = 2\sum_{n\geq 0}(4^n - 1)B_{2n}\frac{z^{2n-1}}{(2n)!}$$

and

$$\frac{d}{dz}\tanh(z/2) = \frac{2e^z}{(e^z + 1)^2} = 2\sum_{n\geq 0}\frac{4^{n+1} - 1}{2(n+1)}B_{2n+2}\frac{z^{2n}}{(2n)!}.$$

Also, since

$$\cosh(z) = \frac{e^z + e^{-z}}{2} = \sum_{n\geq 0} \frac{z^{2n}}{(2n)!}$$

then

$$\begin{aligned}(1+\cosh(z))\frac{d}{dz}\tanh(z/2) &= (1+\frac{e^z+e^{-z}}{2})(\frac{2e^z}{(e^z+1)^2}) \\ &= \frac{(e^z+1)^2}{2e^z}\frac{2e^z}{(e^z+1)^2} = 1. \qquad (11)\end{aligned}$$

Finally, we form the binomial convolution of the generating functions for $\cosh(z)$ and $\frac{d}{dz}\tanh(z/2)$:

$$\begin{aligned}\cosh(z)\frac{d}{dz}\tanh(z/2) &= 2\sum_{n\geq 0}\sum_{k=0}^{n}\frac{1}{(2n-2k)!}\frac{4^{k+1}-1}{2k+2}\frac{1}{(2k)!}(2n)!B_{2k+2}\frac{z^{2n}}{(2n)!} \\ &= 2\sum_{n\geq 0}\sum_{k=0}^{n}\binom{2n}{2k}\frac{4^{k+1}-1}{2k+2}B_{2k+2}\frac{z^{2n}}{(2n)!} \\ &= -\frac{d}{dz}\tanh(z/2) = -2\sum_{n\geq 0}\frac{4^{n+1}-1}{2n+2}B_{2n+2}\frac{z^{2n}}{(2n)!}\end{aligned}$$

by (11). Consequently, we have

$$\sum_{k=0}^{n}\binom{2n}{2k}\frac{4^{k+1}-1}{2k+2}B_{2k+2} = -\frac{4^{n+1}-1}{2n+2}B_{2n+2}$$

which implies

$$\sum_{j=1}^{n-1}\binom{2n-2}{2j-2}\frac{4^j-1}{2j}B_{2j} = -\frac{4^n-1}{n}B_{2n}$$

as desired. □