The Extraordinary Bordism Homology

Max Hopkins

December 15, 2016

1 Introduction

Singular homology can often seem too algebraic for the student looking for topological intuition, which is why textbooks and classes often start with the more geometric simplicial homology. For those who have had experience with manifolds and differential topology, a few texts also offer an explanation of Unoriented bordism homology, proffering this as a more intuitive parallel to singular homology [2]. However, perhaps due to the fact that it is presented as a more palatable, intuitive version of singular homology, most sources seem skim or completely omit the proofs necessary to show that Bordism does indeed give rise to a homology theorem, that is the statements and proofs of the Eilenberg-Steenrod axioms. In fact, bordism homology is not a full homology theory: it fails the 7th Eilenberg-Steenrod axiom of dimension, making it an extraordinary homology theory. This is covered in Conner's Differentiable Periodic Maps for the oriented variant of bordism homology, but in my search I have found no sources which compile a full list of the axioms for the simpler unoriented version. In this paper, my goal is to briefly introduce the concept of unoriented bordism to the reader, prove the first 6 Eilenberg-Steenrod axioms, and show why the 7th fails. Note that because this paper focusses on only a basic understanding of bordism and proving it is an extraordinary homology theory, we will take results from differential topology for granted when necessary, and state when this is the case.

2 Cobordism

Cobordism quite literally means "together boundary," and the formal definition is not far from this etymological one:

Definition 1. *For closed k-manifolds M_1, M_2, we say they are cobordant if there exists a diffeomorphism*

$$i : M_1 \amalg M_2 \to \partial N$$

where N is a k+1 manifold with boundary. N is known as a bordism of M_1 and M_2

As one can tell immediately from the definition, at least in lower dimensions this is a very intuitive concept. For instance: $M \times [0,1]$ is a bordism of M with itself for any closed manifold. This is best visualized as the cylinder for $M = S^1$.

There is an alternate convention for defining cobordism, and this is through the idea of a null-bordism. For this definition, we will need to consider $N = \emptyset$ to be a manifold of any dimension.

Definition 2. *A closed manifold M is null-bordant if it is cobordant to $K = \emptyset$.*

Of course, by definition this only occurs when M is the boundary of some N. To the reader familiar with homology, this should immediately hint towards the idea of modding by the image of the boundary map, as $K = \emptyset$ and $M = \partial N$ are in the same equivalence class for all N. Now here we are getting a bit ahead of ourselves, first we wish to show that cobordism is an equivalence relation.

Proposition 3. *Cobordism is an equivalence relation*

Proof. We will sketch transitivity, since the reflexivity and symmetry are immediate. Let M_1, M_2, and M_3 be closed k-manifolds s.t. $M_1 \sim M_2$, and $M_2 \sim M_3$. The idea behind the proof is to "glue" the cobordisms for these relations N and N' at the side of their boundary which is given by M_2. Assuming we may glue N and N' as such and keep the smooth structure (this indeed the case), the resulting manifold N'' has boundary diffeomorphic to $M_1 \amalg M_3$, and is thus a cobordism of M_1 and M_3, so $M_1 \sim M_3$ as desired. □

Now there is an obvious way to create a group structure from this definition, and that is in considering the cobordism classes of the set of closed n-manifolds up to diffeomorphism (MO_n) with the operation given by disjoint union. This is called the unoriented Thom bordism group after Thom's seminal work on bordism theory [1]. These groups are all known [3], and as we will discuss in the next section it turns out that bordism homology reduces to this when working over the single point space, and thus fails to satisfy all of the Eilenberg-Steenrod axioms.

3 Bordism Homology

In a way, the above construction is similar to simplicial homology in that the elements of MO_n are equivalence classes of spaces "mod boundary", particularly manifolds, just as elements of simplicial homology are cycles mod boundary made out of simplices. As in the transition to singular homology, where one looks at continuous maps from simplices into a topological space X, in bordism homology one examines continuous maps from compact n-manifolds into X. We will define a Singular Manifold over X to be a pair (M, f) such that $f : M \to X$ is a continuous map. Further we define the boundary map $\partial(M, f) = (\partial M, f|\partial M)$. We will also use the notation $f|\partial M = \partial f$ for a singular manifold (M, f) throughout. Now there are a couple ways to define a cobordism over X from here. Hopkins, in his notes on bordism homology, states that

Definition 4. *two singular n-manifolds $(f_1, M_1), (f_2, M_2)$ are cobordant if there exists an $n+1$ manifold (h, N) and a diagram*

$$\begin{array}{ccc} M_1 \amalg M_2 & \xrightarrow{\quad g \quad} & \partial N \\ & {\scriptstyle f_1 \amalg f_2} \searrow \quad \swarrow {\scriptstyle \partial h} & \\ & X & \end{array}$$

where g is a diffeomorphism.

A singular manifold is null-bordant then if it is cobordant to $\emptyset$. Dieck chooses to define the null-bordance first, stating

Definition 5. *A null bordism of the closed singular manifold (M, f) over X is a triple (B, F, ϕ) of a singular manifold (B, F) and a diffeomorphism $\phi : \partial M \to \partial B$ such that $(F|\partial B) \circ \phi = f$*

The two definitions give the same diagram and diffeomorphism, so the reader may choose whichever is easier to follow. To show that this idea is not far from the above, we provide the same example:

Example 6. *Any closed singular n-manifold (M, f) over X is cobordant to itself.*

Proof. Consider the singular $(n+1)$-manifold $(M \times [0,1], F)$ over X, where $F(x,t) = f(x)$. Then we have a diagram (for simplicity let $B = M \times [0,1]$)

$$\begin{array}{ccc} M \amalg M & \xrightarrow{\quad i \quad} & M \times \{0\} \amalg M \times \{1\} \\ & {\scriptstyle f \amalg f} \searrow \quad \swarrow {\scriptstyle F|\partial B} & \\ & X & \end{array}$$

where i is simply the diffeomorphism sending the first copy of M to $M \times \{0\}$ and the second copy of M to $M \times \{1\}$ □

Now similar to above, bordism is an equivalence relation. Here the idea relies on additional differential topology, so we omit the proof, one may find it in most discussions of bordism homology. Then, adopting Dieck's notation, we may define the nth dimensional bordism homology of X $N_n(X)$ to be the set of closed singular n-manifolds over X up to bordism. As it turns out, this is an abelian group whose elements are at most order 2 The constant map serves as the identity, and every manifold is cobordant to itself, so the above is clear. With these out of the way, it seems natural to address the map induced by a continuous map of topological spaces–a concept integral to homology.

Proposition 7. *a continuous map $f : X \to Y$ induces a homomorphism $f_* : N_n(X) \to N_n(Y)$*

Proof. For those familiar with singular homology, the proof is similar. Given a bordism class $[(M_1, \sigma)]$ of X, we define $f_*[(M_1, \sigma)] = [f(M_1, \sigma)] = [(M_1, f\sigma)]$ and check this is well defined. We must show that $f(\sigma)$ is cobordant to $f(\sigma')$ for any $\sigma' : M_2 \to X$ cobordant to σ. Using the diagram given by this cobordance and precomposing with f, we get the diagram

$$\begin{array}{ccc} M_1 \amalg M_2 & \xrightarrow{\quad g \quad} & \partial N \\ & {\searrow}^{f(\sigma)\amalg f(\sigma')} \quad {\swarrow}_{f(\partial h)} & \\ & Y & \end{array}$$

where we note $f(\sigma \amalg \sigma') = f(\sigma) \amalg f(\sigma')$ by definition. This diagram commutes as we have $\partial h \circ g = \sigma \amalg \sigma'$, but then $f(\partial h) \circ g = f(\sigma) \amalg f(\sigma')$ so $f(\sigma)$ is cobordant to $f(\sigma')$ as desired. □

4 Relative Bordism Homology

Now before tackling the Eilenberg-Steenrod axioms, we will need to define bordism homology over a pair of spaces (X, A). This is a necessity for any homology theory–those who know singular homology will recognize this as the homology given by the chain complex $(\frac{C_n(X)}{C_n(A)}, \partial)$, where ∂ is the induced boundary map on the quotient space. The transition to relative homology here will not be quite as smooth, and we will need redefine both our objects of interest and bordism itself. Let a singular manifold over the pair (X, A) be a compact manifold M, and a map of pairs $f : (M, \partial M) \to (X, A)$. Bordism now takes a slightly more complex form:

Definition 8. *(M_0, f_0) and (M_1, f_1) are cobordant if there exists a pair $(n+1)$-manifold (B, F) such that (1) ∂B is the union of M_0, M_1, and M', (2) $\partial M' = \partial M_0 \amalg \partial M_1$ and $M_i \cap M' = \partial M_i$, (3) $F|M_i = f_i$, and (4) $F(M') \subset A$. Note that (B, F) is not a singular manifold! This would already give the condition $F(M') \subset A$, but would restrict $f(M_0)$ and $f(M_1)$ to A which we cannot allow.*

This looks much the same as above but with the extra M' term and added conditions. This comes from the fact that we are no longer working over closed manifolds: M' gives an added bordism for the boundaries of M_0 and M_1 which we will need for the boundary map on the long exact sequence of pairs to be well defined. Once again we have that bordism is an equivalence relation (again the proof relies heavily on differential topology so we omit it, see Dieck 15.10.3). Let the set of compact singular manifolds over (X, A) up to bordism be denoted $N_n(X, A)$. All results above follow similarly: this is a group under disjoint union, elements are of order at most 2, and a map of pairs $f : (X, A) \to (Y, B)$ induces a map $f_* : N_n(X, A) \to N_n(Y, B)$.

The last definition we need regarding the relative groups is the boundary map $\partial_* : N_n(X, A) \to N_{n-1}(A)$ which will appear in the LES of pairs and the naturality axiom.

Remark 9. *For a singular n-manifold (M, f) over (X, A), $\partial(M, f) = (\partial M, f|\partial M)$, is a well-defined homomorphism from $N_n(X, A) \to N_{n-1}(A)$.*

Proof. The target follows from the fact that ∂M is a closed $(n-1)$-manifold, and $f|\partial M$ maps exclusively into A by definition of the (M, f). Further this is well defined due to M' as mentioned above. That is, if $[M_0, f_0] = [M_1, f_1]$, then for a bordism of the two B, $(M', F|M')$ as defined above gives a bordism of $(\partial M_0, \partial f_0)$ and $(\partial M_1, \partial f_1)$ as $F|M_i = f_i$ and $\partial M = \partial M_0 \amalg \partial M_1$ □

With that, we are finally ready to present the 7 axioms. Instead of in numerical order, we present the axioms by length of proof, unsurprisingly leaving the long exact sequence of a pair for last.

5 The Eilenberg-Steenrod Axioms

5.1 Axioms 1-3

Now that we have fully defined Relative bordism homology, axioms 1-3 will follow immediately but we provide proofs for completeness:

Remark 10. *The identity $I : (X, A) \to (X, A)$ on a pair induces the identity $I_* : N_n(X, A) \to N_n(X, A)$*

Proof. $I_*[M, f] = [(M, I \circ f)] = [M, f]$ □

Remark 11. *for $f : (X, A) \to (Y, B)$ and $g : (Y, B) \to (Z, C)$, $(fg)_* = f_* g_*$*

Proof. By definition we have $(fg)_*[M, h] = [M, f(gh)] = f_*[(M, gf)] = f_* g_*[M, f]$ □

Remark 12. *The boundary map ∂_* is natural, that is for a map $\phi : (X, A) \to (Y, B)$, the diagram*

$$\begin{array}{ccc} N_n(X,A) & \xrightarrow{\delta_*} & N_{n-1}(A) \\ \downarrow{\phi_*} & & \downarrow{\phi_*|A} \\ N_n(X,B) & \xrightarrow{\delta_*} & N_{n-1}(B) \end{array}$$

commutes.

Proof. $\phi_* \partial_*[M, f] = [\phi(\partial M, f|\partial M)] = [\partial M, \phi(f|\partial M)]$
$\partial_* \phi_*[M, f] = [\partial(M, \phi f)] = [\partial M, (\phi f)|\partial M]$ □

5.2 Axiom 5: Homotopy

Now we will move on to the first non-trivial axiom, which provides a nice transition into a method we will continue to use, proof by explicit construction of a bordism. The idea for the homotopy axiom is fairly obvious–we must construct a bordism using the given homotopy:

Proposition 13. *Given homotopic maps of pairs f and $g : (X, A) \to (Y, B)$, $f_* = g_*$*

Proof. For any singular n-manifold $(M, \phi) \in N_n(X, A)$, we wish to exhibit a bordism between $f_*(M, \phi) = (M, f\phi)$ and $g_*(M, \phi) = (M, g\phi)$. Now we know there exists a homotopy of f and g, $h : (X \times I, A \times I) \to (Y, B)$. This allows us to define a singular $(n+1)$-manifold, $(M \times I, F)$, where $F(x, t) = h(\phi(x), t)$. The boundary of $M \times I$ is $M \times \partial[0, 1] \amalg \partial M \times [0, 1]$, and further $F(x, 0) = f\phi(x)$ and $F(x, 1) = g\phi(x)$. Then this singular manifold satisfies (1) and (3) of the above definition immediately, note for (1) that $M' = \partial M \times [0, 1]$. (2) follows from having defined M' as such. For (4), we must examine $F(\partial M \times [0, 1]) = h(\phi(\partial M), [0, 1])$. We know $\phi(\partial M) \subset A$, and because h is a homotopy of pairs, $A \times [0, 1]$ always maps into B, thus we have $F(M') \subset B$ as desired, giving us a bordism of $f_*(M, \phi)$ and $g_*(M, \phi)$. □

5.3 Axiom 6: Excision

The proof for excision requires a deeper foray into differential topology not within the scope of this paper. Thus we will state a key lemma necessary for the result and omit the proof, see Conner and Floyd section 1.3 for a full discussion.

Lemma 14. *Let M_0 and M_1 be closed, disjoint submanifolds of a compact manifold M. Then there exists a submanifold M' closed in M such that $M_0 \subset M'$, and $M_1 \cap M' = \emptyset$ which may be given a differentiable structure [1]*

Now the proof for excision will also rely on a sort of analog of the theorem over the manifolds themselves:

Lemma 15. *Given a closed singular n-manifold (M, f) over (X, A) and a compact sub-manifold V such that $f : (M, M \backslash V) \to (X, A)$ is a map of pairs, then $[M, f] = [V, f|V]$ in $N_n(X, A)$*

Proof. We will construct a bordism of (M, f) and $(V, f|V)$. Consider the singular $(n+1)$-manifold $(M \times I, F)$, such that $F(x,t) \to f(x)$. Its boundary is two disjoint copies of M, i.e. $M \times \partial I$ (as M is closed), so we consider the submanifold $V \times 1 \cup M \times 0$. The complement of this submanifold is $M \backslash V \times 1$, and $F(M \backslash V \times 1) = f(M \backslash V) \subset A$ by definition. Now we prove this gives a bordism: relabeling M as $M \times 0$, V as $V \times 1$, and $M' = M \backslash V \times 1$, gives the desired relation. $\partial M' = \emptyset$ and $M' \cap V \times 1 = M' \cap M \times 0 = \emptyset$, so (1) and (2) are satisfied. Our definition of F immediately gives $F|M \times 0 = f$ and $F|V \times 1 = f|V$, and we showed above that $F(M') \subset A$ so we have satisfied (3) and (4) and are done. □

Proposition 16. *The (weak) excision property holds for Relative bordism homology: the inclusion i induces an isomorphism $i_* : N_n(X \backslash Z, A \backslash Z) \to N_n(X, A)$ for $\bar{Z} \subset Int(A)$*

Proof. We begin with a singular n-manifold (M, f) in (X, A). For some $Z \subset A \subset X$ whose closure lies in the interior of A, define compact sub-manifolds $M_0 = f^{-1}(X \backslash Int(A))$ and $M_1 = f^{-1}(\bar{Z})$. By Lemma 14, there exists a compact sub-manifold M' with $M_0 \subset M'$ and $M_1 \cap M' = \emptyset$. Now $[M', f|M'] \in N_n(X \backslash Z, A \backslash Z)$, as we have strictly avoided the entire pre-image of the closure of Z. Further, we have by definition that $f(M \backslash M') \subset A$, but then Lemma 15 gives that $i_*[M', f|M'] = [M, f]$ proving that i_* is a surjection.

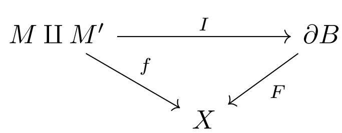
Injection will follow in a similar manner. Assume we have $i_*[M_0, f] = i_*[M_1, g]$, then there exists a bordism (B, F) of (M_0, f) and (M_1, g) over (X, A). Using the same process as above, let $P = F^{-1}(X \backslash Int(A))$, $Q = F^{-1}(\bar{Z})$. Then by Lemma 14 there exists a sub-manifold B' of B such that $P \subset B'$, $B' \cap Q = \emptyset$, and $i_*[B', F|B'] = [B, F]$. We claim $(B', F|B')$ is a bordism of (M_0, f) and (M_1, g). Now Lemma 14 guarantees that B' is closed in B, and thus $\partial B' = \partial B$. Then $(B', F|B')$ inherits conditions (1) and (2) directly from B. Similarly, property (3) is inherited from F, that is $F(M_0) = f$ and $F(M_1) = g$. Lastly condition (4) is satisfied by construction, as $B' \cap F^{-1}(\bar{Z}) = \emptyset$, and $F(M') \subset A$ (due to being a bordism over (X,A)) together imply $F|B'(M') \subset A \backslash Z$ as desired. Then we have constructed a bordism of (M_0, f) and (M_1, f) so $[M_0, f] = [M_1, g]$ and we have proved injectivity. □

5.4 Axiom 4: LES of Pairs

Finally we reach the last axiom bordism satisfies. Because most of our groups are over a single space, we will often be dealing with closed singular manifolds. Thus we will need both the Lemma 15 and another brief result on closed manifolds:

Lemma 17. *Any null-bordism of a closed singular n-manifold (M, f) over (X, A) is a bordism of (M, f) and some (M', F) over X such that $F(M') \subset A$*

Proof. Let (B, F) be a null-bordism of a closed singular n-manifold (M, f) over (X, A), then by definition $\partial B = M \cup M'$, $F|M = f$, $M \cap M' = \partial M = \emptyset$, and lastly $F(M') \subset A$. Then we have the diagram



with the added condition $F(M') \subset A$ as desired □

With this out of the way, we may show how to construct the long exact sequence for a pair (X, A). A terse proof is offered in Dieck, but we aim to add significant clarification.

Proposition 18. *The sequence of pairs:*

$$\begin{array}{l} \cdots \longrightarrow N_{n+1}(X,A) \xrightarrow{\partial_*} \\ N_n(A) \xrightarrow{i_*} N_n(X) \xrightarrow{j_*} N_n(X,A) \ldots \xrightarrow{\partial_*} \\ \ldots N_0(A) \xrightarrow{i_*} N_0(X) \xrightarrow{j_*} N_0(X,A) \longrightarrow \\ 0 \end{array}$$

is a long exact sequence where $i : A \to X$ is the standard inclusion, j_* takes a closed singular manifold $(M, f) \in N_i(X)$ to $(M, f) \in N_i(X, A)$ where $\partial M = \emptyset$, and δ_* is boundary map defined above.

Proof. We begin with proving exactness at $N_n(A)$. Consider $i_* \circ \partial_*[M, f] = i_*[\partial M, f|\partial M] = [i(\partial M, f|\partial M)] = [\partial M, f|\partial M] \in N_n(X)$. (M, f) gives an obvious null bordism of $(\partial M, f|\partial M)$, so this is $[\emptyset] = 0$. Similarly, consider any element $(M, f) \in N_n(A)$ in the kernel of i_*. By definition this simply means that $i(M, f) = (M, f)$ over X is null-bordant. Then there exists some null-bordism (B, F) of (M, f). Being a bordism over X, we know ∂B is diffeomorphic to M by some g, and $\partial F \circ g = f$, but then $(\partial B, F|\partial B)$ is cobordant to (M, f), as the diagram below commutes, and we get $[M, f] = \partial[B, F]$ as desired.

$$\begin{array}{ccc} M \amalg \partial B & \xrightarrow{I \amalg g} & M \times \partial(I) \\ & {\scriptstyle f \amalg \partial F} \searrow \quad \swarrow {\scriptstyle f \amalg f} & \\ & A & \end{array}$$

Now we examine exactness at $N_n(X)$. For some element $[M, f] \in N_n(A)$, we have by definition $j_* \circ i_*[M, f] = [M, f] \in N_n(X, A)$ where $\partial M = \emptyset$, and most importantly $f(M) \subset A$. This last condition allows us to apply the Lemma 15 with $V = \emptyset$! Then we have $j_* i_*[M, f] = [\emptyset] \in N_n(X, A)$ as desired.

Now take some element $[M, f] \in N_n(X)$ in the kernel of j_*, i.e. $[M, f] \in N_n(X, A) = 0$. Like above, this means there exists a null-bordism (B, f) of (M, f). By Lemma 17, (B, f) is a cobordism of (M, f) and (M', F) for some M' such that $F(M') \subset A$, but then there exists $[M', F] \in N_n(A)$ such that $i_*[M', F] = [M', F] = [M, f] \in N_n(X)$

Finally we move to proving exactness at $N_n(X, A)$. Given $[M, f] \in N_n(X)$, consider $j_* \partial_*[M, f] = [\partial j(M, f)] = [\partial M, f|\partial M]$ but M is closed so this is $[\emptyset] = 0$. Now consider an element $[M, f]$ in the kernel of ∂_*. As above pick a null-bordism (B, F) of $(\partial M, f|\partial M)$. This is a null-bordism over A, so its boundary is diffeomorphic to ∂M. Then because M and B have diffeomorphic boundaries, we may glue them along the boundary to create a closed manifold (N, h). Note $F|\partial M = f$, so setting $h = f$ on M and F on B is well-defined and continuous. Now consider $j_*[N, h] \in N_n(X, A)$ and recall that $B \subset N$ was a null-bordism over A and thus $h(int(B)) \subset A$. Then by Lemma 15 we have $j_*[N, h] = j_*[N \backslash int(B), h|N \backslash int(B)] = [M, f]$ □

5.5 Axiom 7: Dimension

Despite its simplicity, we have saved axiom 7 for last. Indeed, as the reader may recall, this is because bordism homology actually fails the Dimension axiom. As we mentioned above, $N_n(pt) = MO_n$, the unoriented Thom bordism group.

Remark 19. $N_n(pt) = MO_n$

Proof. We define the obvious homomorphism $i : N_n(pt) \to MO_n$ s.t. $i[M, c] \to [M]$ where c is the constant map. Surjection follows immediately from definition. Assume $i([M_0, c]) = ([M_1, c])$, then M_0 and M_1 are cobordant, so there exists a compact $(n+1)$-manifold N with a diffeomorphism $i : \partial N \to M_0 \amalg M_1$, which gives the bordism

$$\begin{array}{ccc} M_0 \amalg M_1 & \xrightarrow{\quad I \quad} & \partial N \\ & {\scriptstyle c}\searrow \quad \swarrow{\scriptstyle c} & \\ & X & \end{array}$$

and thus $[M_0, c] = [M_1, c]$. $\square$

Of course we are note quite done, as we have not proven $\exists\, n > 0$ s.t. $MO_n \neq 0$, but as mentioned above these groups are completely known. The simplest example comes from the fact that $\mathbb{R}P^2$ is a closed manifold but not the boundary of any compact 3-manifold, and thus $MO_2 \neq 0$. In fact, $MO_2 = \mathbb{Z}/2$, and is completely generated by this bordism class [3]

6 Conclusion

We have introduced the topic of bordism and singular manifolds in a hopefully intuitive fashion for the reader, and given straight-forward proofs for the first 6 Eilenberg-Steenrod axioms, as well as showing why the 7th fails. In this, we have shown that bordism homology is an extraordinary homology theory. For more information on the differential background to bordism we omitted from this paper, one should see the opening sections of Conner's Differential Periodic Maps.

References

[1] Conner, P. (1979). Differentiable Periodic Maps : Second Edition (Lecture Notes in Mathematics ; 738).

[2] Dieck, T. (2008). Algebraic Topology : Corrected 2nd printing, 2010 (EMS Textbooks in Mathematics (ETB)). Zuerich, Switzerland: European Mathematical Society Publishing House.

[3] Hopkins, M. J. (2016). Bordism Homology and Surfaces