Graphons, mergeons, and so on!

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with

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THE OHIO STATE UNIVERSITY
theory of machine learning
theory of machine learning

classification

clustering
theory of machine learning

classification

clustered
theory of machine learning

classification $\gg$ clustering
What is the correct clustering?

- In general, there is no single answer.
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- But consider a statistical approach...
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What is the correct clustering?

- In general, there is no single answer.
- But consider a statistical approach...

In the statistical approach, there is often a natural ground truth clustering.
Example: the **density** model.

0. Model the data as coming from a **probability density**.
Example: the **density** model.

1. Define the **clusters** of the **density**.
   - Region of **high probability**.
Example: the density model.

1. Define the clusters of the density.
   - Connected component of \( \{ f \geq \lambda_1 \} \)?
Example: the density model.

1. Define the clusters of the density.
   ▶ Connected component of \( \{ f \geq \lambda_2 \} \)?
Example: the density model.

1. Define the clusters of the density.
   ▶ Connected component of \( \{ f \geq \lambda_3 \} \)?
Example: the density model.

1. Define the clusters of the density.
   - Connected component of \( \{ f \geq \lambda \} \) for any \( \lambda > 0 \).
Example: the density model.

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1. Define the clusters of the density.
   - Elements of the density cluster tree of \( f \).
Example: the density model.

1. Define the clusters of the density.
   ▶ Elements of the density cluster tree of \( f \).

Natural goal of clustering in the density model: 
Recover the density cluster tree.
Example: the density model.

2. Develop a notion of convergence to the density cluster tree.

Sample $n$ points from density.
Example: the density model.

2. Develop a notion of convergence to the density cluster tree.

Apply hierarchical clustering algorithm.
Example: the density model.

2. Develop a notion of convergence to the density cluster tree.

As $n \to \infty$...
Example: the density model.

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As \( n \to \infty \)...
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2. Develop a notion of convergence to the density cluster tree.
     ▷ Clusters disjoint in true tree should be disjoint in clustering.
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2. **Develop a notion of convergence to the density cluster tree.**
     - Clusters disjoint in true tree should be disjoint in clustering.

3. **Construct consistent density clustering algorithms.**
   - Hartigan consistent:
     - Robust single linkage (Chaudhuri & Dasgupta, 2010)
     - Tree pruning (Kpotufe & von Luxburg, 2011)
Example: the density model.

2. **Develop a notion of convergence to the density cluster tree.**
     - Clusters disjoint in true tree should be disjoint in clustering.
     - Pairs of points merge around same height in both trees.

3. **Construct consistent density clustering algorithms.**
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Example: the density model.

2. Develop a notion of convergence to the density cluster tree.
     ▶ Clusters disjoint in true tree should be disjoint in clustering.
   ▶ Strong notion: Merge distortion (EBW, 2015).
     ▶ Pairs of points merge around same height in both trees.

3. Construct consistent density clustering algorithms.
   ▶ Hartigan consistent:
     ▶ Robust single linkage (Chaudhuri & Dasgupta, 2010)
     ▶ Tree pruning (Kpotufe & von Luxburg, 2011)
   ▶ Consistent in merge distortion:
     ▶ (EBW, 2015)
In this talk, we develop a statistical theory of graph clustering:

0. We model the data as coming from a graphon.
1. We define the clusters of a graphon.
2. We develop a notion of convergence to the graphon’s clusters.
3. We provide a clustering algorithm which converges to the graphon’s clusters.
In this talk, we develop a **statistical theory of graph clustering**:

0. **We model** the data as coming from a graphon.
1. **We define** the clusters of a graphon.
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3. **We provide** a clustering algorithm which **converges** to the graphon’s clusters.
Background: the stochastic blockmodel.

- Much of existing theory is in the stochastic blockmodel.
- This is a model for generating random graphs.
- Each node belongs to one of $k$ blocks, or communities.
- Edge probabilities parameterized by symmetric $k \times k$ matrix $P$:
  - Prob. of edge within community $i$ given by $P_{ii}$.
  - Prob. of edge between communities $i$ and $j$ given by $P_{ij}$.
- Example: 2-block model.
  - Social network of girls and boys at a school.
Sampling from a blockmodel.

We can generate a random graph with $n$ nodes from $P$ as follows...

1. Sample communities uniformly with replacement.
Sampling from a blockmodel.

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1. Sample communities uniformly with replacement.

$$\begin{bmatrix}
\text{BBB} & \text{BBB} & \text{BBB} & \text{BBB} \\
\text{BCC} & \text{BCC} & \text{BCC} & \text{BCC} \\
\text{BBB} & \text{BBB} & \text{BBB} & \text{BBB} \\
\text{BCC} & \text{BCC} & \text{BCC} & \text{BCC} \\
\end{bmatrix}$$
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1. Sample communities uniformly with replacement.
2. Sample edges with probability according to \( P \).
Sampling from a blockmodel.

We can generate a random graph with $n$ nodes from $P$ as follows...

1. Sample communities uniformly with replacement.
2. Sample edges with probability according to $P$.

Add edge with probability $P$. 

Add edge with probability $P$. 

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Add edge with probability $P$. 

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Add edge with probability $P$. 

Add edge with probability $P$.
Sampling from a blockmodel.

We can **generate** a random graph with $n$ nodes from $P$ as follows...

1. Sample **communities** uniformly with **replacement**.
2. Sample edges with **probability** according to $P$.

Add edge with probability $P$. 
Sampling from a blockmodel.

We can generate a random graph with $n$ nodes from $P$ as follows...

1. Sample communities uniformly with replacement.
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We can generate a random graph with \( n \) nodes from \( P \) as follows...

1. Sample communities uniformly with replacement.
2. Sample edges with probability according to \( P \).

Repeat for all pairs of nodes.
Sampling from a blockmodel.

We can generate a random graph with $n$ nodes from $P$ as follows...

1. Sample communities uniformly with replacement.
2. Sample edges with probability according to $P$.
3. Forget community labels.
Equivalent parameterizations.

Permuting the rows/columns of $P$ does not change graph distribution.
Clustering theory in the stochastic blockmodel.

1. Define the clusters of the blockmodel.
   - The communities used to define the blockmodel.

2. Develop a notion of convergence to the communities.
   - Recover community labels exactly as $n \to \infty$.

3. Construct consistent blockmodel clustering algorithms.
   - Spectral methods, such as (McSherry, 2001).
Problem: Many real-world networks not well-fit by blockmodel.

- Large networks (Facebook, LinkedIn, etc.) are complicated.
- The 2-blockmodel is very simple.
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- Large networks (Facebook, LinkedIn, etc.) are complicated.
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The limit of a blockmodel is...

$$\lim_{k \to \infty} \left\{ \begin{array}{c}
\begin{array}{c}
\text{Block A} \\
\text{Block B} \\
\text{Block C}
\end{array}
\end{array} \right\}, \ldots = ?$$
The limit of a blockmodel is...

$$\lim_{k \to \infty} \left\{ \begin{array}{c} \text{[Blockmodel elements]} \\ \vdots \end{array} \right\}$$

...a graphon!

$$W : [0, 1]^2 \to [0, 1]$$

symmetric, measurable
The limit of a blockmodel is...

\[ \lim_{k \to \infty}^{\dagger} \begin{pmatrix} \begin{array}{ccc} \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \\ \end{array} \end{pmatrix} \], \begin{pmatrix} \begin{array}{ccc} \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \\ \end{array} \end{pmatrix}, \begin{pmatrix} \begin{array}{ccc} \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \\ \end{array} \end{pmatrix}, \begin{pmatrix} \begin{array}{ccc} \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \\ \end{array} \end{pmatrix}, \ldots \]

\[ = \]

...a graphon!

symmetric, measurable

\[ W : [0,1]^2 \to [0,1] \]

\[ \dagger \text{ Convergence in so-called cut metric, (Lovász, 2012).} \]
Interpretation: The adjacency of an infinite weighted graph.
Interpretation: The adjacency of an **infinite** weighted graph.

Graphon “nodes” are points $x, y \in [0, 1]$. 
**Interpretation:** The adjacency of an **infinite** weighted graph.

\[ W(x, y) \] is the weight of the “edge” \((x, y)\).
Sampling a graph from $W$.

Graphon sampling is analogous to sampling from a blockmodel.
Sampling a graph from $W$.

First, sample $n$ graphon nodes, i.e., points from $\text{Unif}[0, 1]$. 
Sampling a graph from $\mathcal{W}$.

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First, sample $n$ graphon nodes, i.e., points from $\text{Unif}[0, 1]$. 

\[ x_1, x_2, x_3 \]
Sampling a graph from $\mathcal{W}$.

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First, sample $n$ graphon nodes, i.e., points from $\text{Unif}[0, 1]$. 

\begin{align*}
x_2 & \quad x_5 \quad x_1 \quad x_4 \quad x_3 \\
x_2 & \quad x_5 \quad x_1 \\
x_4 & \quad x_3 \\
x_3 & \quad x_4 \quad x_5
\end{align*}
Sampling a graph from $W$.

First, sample $n$ graphon nodes, i.e., points from Unif[0, 1].
Sampling a graph from $\mathcal{W}$.

Include edge $(x_1, x_5)$ with probability $W(x_1, x_5)$. 

$\mathcal{W}(x_1, x_5)$
Sampling a graph from $\mathcal{W}$.

By chance, edge $(x_1, x_5)$ is included.
Sampling a graph from $\mathcal{W}$.

Include edge $(x_3, x_6)$ with probability $\mathcal{W}(x_3, x_6)$. 
Sampling a graph from $W$.

By chance, edge $(x_3, x_6)$ is omitted.
Sampling a graph from $W$. 

Repeat for all possible edges.
Sampling a graph from $\mathcal{W}$.

Forget node labels, obtaining **undirected** & **unweighted** graph.
Sampled graphs converge to the graphon they were sampled from.
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A graphon $W$ defines a very rich distribution on graphs.

- Better models real-world data (Hoff, 2002).
- Subsumes many models, e.g., blockmodel:
A graphon $W$ defines a very rich distribution on graphs.

- Better models real-world data (Hoff, 2002).
- Subsumes many models, e.g., blockmodel:

\[
\begin{pmatrix}
p_1 & q \\ q & p_2
\end{pmatrix} = \begin{pmatrix}
p_1 & q \\ q & p_2
\end{pmatrix}
\]

Warning! Graphons can be much more complex than blockmodels.

- Present several unique and subtle technical issues.
Issue 1: A graphon node or edge is not meaningful by itself.

\[
\lim_{k \to \infty} \left( \begin{array}{ccc}
1 & 2 & 3 \\
1 & 1 & 3 \\
1 & 1 & 1 \\
\end{array} \right), 
\left( \begin{array}{ccc}
1 & 2 & 3 \\
1 & 1 & 3 \\
1 & 1 & 1 \\
\end{array} \right), 
\left( \begin{array}{ccc}
1 & 2 & 3 \\
1 & 1 & 3 \\
1 & 1 & 1 \\
\end{array} \right), 
\left( \begin{array}{ccc}
1 & 2 & 3 \\
1 & 1 & 3 \\
1 & 1 & 1 \\
\end{array} \right), \ldots 
\]
Issue 1: A graphon node or edge is not meaningful by itself.

\[
\lim_{k \to +\infty} \left[ \ldots \right], \ldots
\]

In a careful approach:

- Do not reference single nodes/edges in a graphon.
- Only deal with equivalence classes of sets of nodes modulo null sets.

In what follows, we largely ignore the issue in the interest of time and simplicity; see paper for details.
Recall: $P_1$ and $P_2$ define the same stochastic blockmodel if they are equivalent up to relabeling.

Issue 2: Similarly, $W_1$ and $W_2$ define the same graphon model $\iff$ they are equivalent up to relabeling, (Lovász, 2012).
Issue 2: A graphon relabeling can be very complex.

- **A relabeling** is a map $\varphi : [0, 1] \rightarrow [0, 1]$.
- $\varphi$ must be "measure preserving".
  - Only in one direction: preimage.
  - Can map a null set to a set of full measure!
- **Does not** need to be a bijection. Far from it!
Issue 2: A graphon relabeling can be very complex.

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- $\varphi$ must be “measure preserving”.
  - Only in one direction: preimage.
  - Can map a null set to a set of full measure.

There is usually no canonical way to label a graphon.

- For presentation, we will use a “nice” labeling of “nice” graphons; i.e., piecewise constant.
- But our definitions will make sense for any labeling of any graphon; i.e., arbitrarily-complex measurable function.
A statistical theory of graphon clustering.

In this talk...

0. We model the data as coming from a graphon.

We give answers to the following:

1. What are the clusters of a graphon?
2. How do we define convergence to the graphon’s clusters?
   ▶ I.e., statistical consistency.

3. Which clustering algorithms are consistent?
What are the clusters of a graphon?

We interpret the graphon as the adjacency of an infinite weighted graph.
What are the clusters of a graphon?

Each link in this depiction corresponds to a region of the graphon.
What are the **clusters** of a graphon?

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Each link in this depiction corresponds to a region of the graphon.
What are the clusters of a graphon?

- We define clusters to be connected components.
- Use generalization of graph connectivity, extends (Janson, 2008).
- Key: Insensitive to null sets, e.g., single edges.
What are the clusters of a graphon?

- In fact, we can speak of the clusters at various levels.
- Intuitively: three clusters (connected components) at level $\lambda_3$.
- Any pair ($\bigcirc$, $\bigcirc$) are in same cluster at $\lambda_3$. Same for ($\bigcirc$, $\bigcirc$) & ($\bigcirc$, $\bigcirc$).
What are the clusters of a graphon?

▶ In fact, we can speak of the clusters at various levels.
▶ Intuitively: three clusters (connected components) at level $\lambda_3$.
▶ Any pair (○, ○) are in same cluster at $\lambda_3$. Same for (○, ○) & (○, ○).
▶ Naturally encoded as function $M(○, ○) = M(○, ○) = M(○, ○) = \lambda_3$
What are the clusters of a graphon?

- In fact, we can speak of the clusters at various levels.
- Intuitively: red and blue clusters merge at level $\lambda_2$.
- Any pair (●, ○) are in same cluster at $\lambda_2$.
- Naturally encoded as $M(●, ○) = M(○, ●) = \lambda_2$. 

![Diagram showing clusters and their merging levels]

$\lambda_1$ $\lambda_2$ $\lambda_3$

$\lambda_1$ $\lambda_2$ $\lambda_3$ $\lambda_3$

$\lambda_1$ $\lambda_2$ $\lambda_3$ $\lambda_3$

$\lambda_1$ $\lambda_2$ $\lambda_3$ $\lambda_3$

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$\lambda_1$ $\lambda_2$ $\lambda_3$ $\lambda_3$

$\lambda_1$ $\lambda_2$ $\lambda_3$ $\lambda_3$
What are the clusters of a graphon?

- In fact, we can speak of the clusters at various levels.
- All clusters merge at level \( \lambda_1 \).
- Encoded as \( M(\bigcirc, \bigcirc) = M(\bigcirc, \bigcirc) = \lambda_1 \).
We call $M$ the mergeon.
We call $M$ the **mergeon**.

- $M(x, y)$ encodes the first level at which $x$ & $y$ are in same cluster.
- As such, $M$ defines the **ground truth** clustering of a graphon.
- **Note**: Mergeon helps deal with subtle technical hurdles.
A mergeon has hierarchical structure. Clusters from higher levels nest within clusters from lower levels.

We call this structure the graphon cluster tree.
If graphons $W_1$ and $W_2$ are the same up to relabeling, then their mergeons and cluster trees are the same up to relabeling.

Surprisingly non-trivial to show.
A statistical theory of graphon clustering.

1. What is the ground truth clustering of a graphon?
   ▶ The mergeon, or, equivalently, the graphon cluster tree.

2. How do we define convergence?
A statistical theory of graphon clustering.

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2. How do we define convergence?

![Diagram showing a clustering algorithm and a ground-truth cluster tree, linked by convergence as $n \to \infty$.]
The merge distortion

How “close” are $C$ and $C'$?
Intuitively, corresponding pairs of nodes should merge at around the same height in each tree.
The merge distortion

$M(\bullet, \circ)$

Merge heights are encoded in the mergeon.
The merge distortion

Merge heights are encoded in the mergeon.
The merge distortion

\[ |M(\bullet, \circ) - M'(\bullet, \circ)| \] is the difference in merge height of \( \bullet, \circ \).
The merge distortion

We introduce the merge distortion \( d(\mathcal{C}, \mathcal{C}') \): the maximum difference in merge height over all pairs, i.e,

\[
d(\mathcal{C}, \mathcal{C}') = \max_{\mathcal{M}, \mathcal{M}'} |M(\mathcal{M}, \mathcal{M}') - M'(\mathcal{M}, \mathcal{M}')|.
\]
Convergence in merge distortion

We say $\hat{C}_n$ converges in merge distortion to $C$ if $d(C, \hat{C}_n) \to 0$ as $n \to \infty$.

Definition

An algorithm is consistent if its output converges in merge distortion to the graphon cluster tree in probability as $n \to \infty$.

- Consistency $\iff$ disjoint clusters are separated as $n \to \infty$.  

A technical detail...

We imagine that the nodes of the graph correspond to graphon nodes.
A technical detail...

We imagine that the nodes of the graph correspond to graphon nodes. But this correspondence is latent and unrecoverable.
A technical detail...

We imagine that the nodes of the graph correspond to graphon nodes. But this correspondence is latent and unrecoverable.

- Need correspondence to compute merge distortion.
- Solution: Compute distortion for all possible correspondences.
- Set of correspondences which result in large merge distortion shrinks as $n \to \infty$. 
A statistical theory of graphon clustering.

1. What is the ground truth clustering of a graphon?
   - The mergeon, or, equivalently, the graphon cluster tree.

2. How do we define convergence/consistency?
   - Convergence in merge distortion using the mergeon.

3. Which clustering algorithms are consistent?
Estimating edge probabilities.

Suppose we sample a graph from this graphon.
Estimating edge probabilities.

Edges within communities have probability $p$; edges across communities have probability $q$. 
Estimating edge probabilities.

If we knew these edge probabilities we could recover the correct clusters.
Estimating edge probabilities.

But the edge probabilities are unknown and the presence/absence of an edge \((i, j)\) tells us little about its probability, \(P_{ij}\).
Estimating edge probabilities.

But the edge probabilities are unknown and the presence/absence of an edge \((i,j)\) tells us little about its probability, \(P_{ij}\).

Idea: Compute estimate \(\hat{P}\) of edge probabilities from a single graph.
Theorem
If $\|P - \hat{P}\|_{\text{max}} \to 0$ in probability as $n \to \infty$, then single linkage clustering using $\hat{P}$ as the input similarity matrix is a consistent clustering method.
Theorem

If \( \|P - \hat{P}\|_{\text{max}} \to 0 \) in probability as \( n \to \infty \), then single linkage clustering using \( \hat{P} \) as the input similarity matrix is a consistent clustering method.

- There are many recent graphon & edge probability estimators.
- But all consistency results are in mean squared error.
- This is too weak. Need consistency in max-norm.
- We modify and analyze the neighborhood smoothing method of (Zhang et al., 2015) to obtain consistency in max-norm.
Neighborhood smoothing

Given this graph...
Neighborhood smoothing

Given this graph... estimate $P_{ij}$. 

Build a neighborhood $N_i$ of nodes with similar connectivity to that of $i$. 
Neighborhood smoothing

- Average number edges from node in neighborhood $N_i$ to $j$.
- Estimated edge probability: $\hat{p}_{ij} = \frac{2}{6} = \frac{1}{3}$. 
Consistency of neighborhood smoothing.

**Theorem**

*Our modified neighborhood smoothing edge probability estimator for P is consistent in max norm.*

**Corollary**

*Consistent graphon clustering method:*

1. *Estimate edge probabilities with our modified neighborhood smoothing approach.*

2. *Apply single linkage clustering to estimated edge probabilities.*
In summary, we develop a statistical theory of graph clustering in the graphon model:

1. We define the clusters of a graphon.
   - The graphon cluster tree/mergeon.

2. We develop a notion of consistency.
   - Convergence in merge distortion.

3. We provide a consistent algorithm.
   - Modified neighborhood smoothing + single linkage.
Grazie!
GRAZIE!
GRAZIE!
Weak isomorphism

- Any graphon $W$ defines a graph distribution.
- Not uniquely! Many graphons define the same distribution.
- The distribution is uniquely determined up to relabeling of $W$. 
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Definition

A measure preserving transformation (i.e., graphon relabeling) $\varphi : [0, 1] \rightarrow [0, 1]$ is a Lebesgue-measurable function whose preimage preserves measure. That is, $\mu(\varphi^{-1}(A)) = \mu(A)$ for all measurable $A \subset [0, 1]$.

Notation: $W^\varphi(x, y) = W(\varphi(x), \varphi(y))$. 
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$$\varphi(x) = \begin{cases} 
    x + \frac{1}{2} & x \leq \frac{1}{2}, \\
    x - \frac{1}{2} & x > \frac{1}{2}
\end{cases}$$
Weak isomorphism

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**Definition**

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$\varphi(x) = 2x \mod 1$
**Weak isomorphism**

**Definition (Lovász)**

Two graphons $W_1$ and $W_2$ are **weakly isomorphic** if there exist measure preserving transformations $\varphi_1$ and $\varphi_2$ such that $W_1^{\varphi_1} \overset{\text{a.e.}}{=} W_2^{\varphi_2}$.

- $W_1$ and $W_2$ define the same distribution iff they are weakly isomorphic.
Weak isomorphism

Definition (Lovász)
Two graphons $W_1$ and $W_2$ are **weakly isomorphic** if there exist measure preserving transformations $\varphi_1$ and $\varphi_2$ such that $W_1^{\varphi_1} \cong W_2^{\varphi_2}$.

- $W_1$ and $W_2$ define the same distribution iff they are weakly isomorphic.
The clusters of a graphon

1. Collect all subsets of $[0, 1]$ which should be clustered at $\lambda$:

   $$\mathcal{A}_\lambda = \{ A \subset [0, 1] : \mu(A) > 0 \text{ and } A \text{ is connected } \forall \, \lambda' < \lambda. \}$$

2. If $A_1, A_2, A \in \mathcal{A}_\lambda$, and $A_1 \cup A_2 \subset A$, then $A_1, A_2,$ and $A$ should all be in the same cluster at $\lambda$. Consider them equivalent.
   - Define equivalence relation on $\mathcal{A}_\lambda$:
     $$A_1 \sim_{\lambda} A_2 \iff \exists A \in \mathcal{A}_\lambda, A \supset A_1 \cup A_2.$$

   - Read: $A_1$ is clustered with $A_2$ at level $\lambda$.
   - $\sim_{\lambda}$ partitions $\mathcal{A}_\lambda$ into equivalence classes of sets which should be in the same cluster.
The clusters of a graphon

3. Define clusters to be “largest” element of each equivalence class.
   ▶ Subtlety in defining “largest”:
     ▶ Suppose $A \in \mathcal{A}_\lambda/\circ\circ_\lambda$ is such an equivalence class.
     ▶ Let $A$ be any representative from $\mathcal{A}$, let $Z$ be a set of zero measure.
     ▶ $A' = A \cup Z$ is a representative of $\mathcal{A}$.
   ▶ In general there is no representative of $\mathcal{A}$ which strictly contains all other representatives in $\mathcal{A}$
   ▶ We can find reps. which contain every other rep. up to a null set, called the “essential maxima” of $\mathcal{A}$:
     \[
     \text{ess max } \mathcal{A} = \{A \in \mathcal{A} : \forall A' \in \mathcal{A}, \mu(A' \setminus A) = 0\}
     \]
   ▶ The clusters of $\mathcal{W}$ at level $\lambda$ are the essential maxima of each equivalence class:
     \[
     \mathcal{C}_W(\lambda) = \{\text{ess max } \mathcal{A} : \mathcal{A} \in \mathcal{A}_\lambda/\circ\circ_\lambda\}
     \]
Consistent algorithms

- Intuitively, estimating the graphon is related to clustering.
- It suffices to estimate the so-called edge probability matrix.

\[ P : P_{ij} = W(x_i, x_j) \]
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Consistent algorithms

- Intuitively, estimating the graphon is related to clustering.
- It suffices to estimate the so-called **edge probability matrix**.
Sample an adjacency matrix $A$ from $P$:

$W$

$P$

(artificially permuted)

$A$
Sample an adjacency matrix $A$ from $P$:

$W$

$P$

(artificially permuted)

$A$

$A$ is a poor estimate of $P$. 
\[ n = 8 \]
$n = 16$
$n = 32$

Left: $P$

Right: $A$
\[ n = 64 \]
\[ n = 128 \]
$n = 256$
Goal: Compute estimated edge probabilities $\hat{P}$ from $A$.

**Theorem**

If $\|P - \hat{P}\|_{max} \rightarrow 0$ in probability as $n \rightarrow \infty$, then single linkage clustering on $\hat{P}$ is a consistent clustering method.
Edge probability estimation

Goal: Compute estimated edge probabilities $\hat{P}$ from $A$.

Theorem

If $\|P - \hat{P}\|_{max} \to 0$ in probability as $n \to \infty$, then single linkage clustering on $\hat{P}$ is a consistent clustering method.

- We need a suitable estimator $\hat{P}$ of edge probabilities.
- Recently, Zhang et al. (2015) proposed neighborhood smoothing.
Neighborhood smoothing

Given $A$, the adjacency matrix of a sampled graph...
Neighborhood smoothing

Consider a node $i$ and its corresponding column of $A$. 
Neighborhood smoothing

Measure similarity to every other node $j$:
\[ d(i, j) = \max_{k \neq i, j} \left| (A^2)_{ik} - (A^2)_{jk} \right| \]
Neighborhood smoothing

Form neighborhood $N_i$ of nodes most similar to $i$. 
Neighborhood smoothing

Average within neighborhood to estimate edge probability:

\[ \hat{P}_{ij} = \frac{1}{2|N_i|} \sum_{i' \in N_i} A_{i'j} + \frac{1}{2|N_j|} \sum_{j' \in N_j} A_{ij'} \]
Neighborhood smoothing

The result is a smoothed estimate $\hat{P}$ of edge probabilities.