Supplementary material for the paper entitled Existence of globally attracting fixed points of viscous Burgers equation with constant forcing. A computer assisted proof

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1 Introduction

This file contains supplementary material for the paper entitled *Existence of globally attracting fixed points of viscous Burgers equation with constant forcing. A computer assisted proof* [C]. We present hereafter elements that have been excluded from the original paper, because we found them standard and/or tedious.

2 The viscous Burgers equation, derivation of a suitable form

As the viscous Burgers equation we consider following PDE

$$\frac{\partial u}{\partial t} + u \cdot \frac{\partial u}{\partial x} - \nu \triangle u = 0 \quad \text{in } \Omega, \quad t > 0,$$

where ν is a positive *viscosity constant*. The equation was proposed by Burgers (1948) as a mathematical model of the turbulence. Later on it has been successfully showed that the Burgers equation models certain gas dynamics (Lighthill (1956)) and acoustic (Blackstock (1966)) phenomena, see [CHQZ] and [Wh]. For our purposes we define this equation on the real line $\Omega := \mathbb{R}$, add *a constant in time forcing* f to the right-hand side and consider initial value problem with periodic boundary conditions

$$u\colon \mathbb{R} \times [0, \mathcal{T}) \to \mathbb{R},$$

$$f\colon \mathbb{R} \to \mathbb{R},$$

$$u_t + u \cdot u_x - \nu u_{xx} = f(x), \tag{1a}$$
$$u(x,0) = u_0(x), \quad x \in \mathbb{R}, \tag{1b}$$
$$u(x,t) = u(x + 2k\pi, t), \quad x \in \mathbb{R},\ t \in [0, \mathcal{T}),\ k \in \mathbb{Z}, \tag{1c}$$
$$f(x) = f(x + 2k\pi), \quad x \in \mathbb{R},\ k \in \mathbb{Z}. \tag{1d}$$

2.1 The viscous Burgers equation in the Fourier basis

In this section we rewrite (1a) using the Fourier basis of 2π periodic functions $\{e^{ikx}\}_{i\in\mathbb{Z}}$. From now on we assume that all functions we use are sufficiently regular to be expanded in the Fourier basis and all necessary Fourier series converge.

Definition 2.1. Let $u(x)\colon \mathbb{R} \to \mathbb{R}$ be a 2π periodic function. We refer to $\{a_k\}_{k\in\mathbb{Z}}$ as the Fourier modes of u. Where $a_k \in \mathbb{C}$ satisfies

$$a_k = \frac{1}{2\pi}\int_0^{2\pi} u(x)e^{-ikx}\,dx, \tag{2}$$

moreover

$$u(x) = \sum_{k\in\mathbb{Z}} a_k e^{ikx}, \quad x \in \mathbb{R}. \tag{3}$$

Lemma 2.2. *Assume that* $|a_k| \leq \frac{M}{|k|^\gamma}$ *for* $k \in \mathbb{Z}$. *If* $n \in \mathbb{N}$ *is such that* $\gamma - n > 1$, *then the function* $u(x) = \sum_{k\in\mathbb{Z}} a_k e^{ikx}$ *belongs to* C^n. *The series*

$$\frac{\partial^s u}{\partial x^s} = \sum_{k\in\mathbb{Z}} a_k \frac{\partial^s}{\partial x^s} e^{ikx}$$

converges uniformly for $0 \leq s \leq n$.

Now, we shall introduce a Lemma that allows to rewrite convolution in the Fourier domain in more convenient form for numerical calculations. In fact we show that the equality $2u \cdot u_x = (u^2)_x$ is also satisfied in the Fourier domain.

Lemma 2.3. *Let* $u\colon \mathbb{R} \to \mathbb{R}$ *be a* 2π *periodic function,* $\{a_k\}_{k\in\mathbb{Z}}$ *are the Fourier coefficients of* u. *Then for all* $k \in \mathbb{Z}$ *we have*

$$2\sum_{m\in\mathbb{Z}} (k-m)(a_{k-m} \cdot a_m) \;=\; 2\sum_{m\in\mathbb{Z}} m(a_{k-m} \cdot a_m) \;=\; k\sum_{m\in\mathbb{Z}} (a_{k-m} \cdot a_m). \tag{4}$$

Proof. We use the substitution $k - m = n$, $n \in \mathbb{Z}$ and symmetry of scalar product in $\mathbb{C}$

$$\sum_{m\in\mathbb{Z}} (k-m)(a_{k-m} \cdot a_m) = \sum_{n\in\mathbb{Z}} n(a_n \cdot a_m) = \sum_{m\in\mathbb{Z}} m(a_m \cdot a_n),$$

$$k \sum_{m\in\mathbb{Z}} (a_{k-m} \cdot a_m) = \sum_{m\in\mathbb{Z}} (k-m)(a_{k-m} \cdot a_m) + \sum_{m\in\mathbb{Z}} m(a_{k-m} \cdot a_m). \quad \blacksquare$$

Lemma 2.4. *Let u_0 be an initial value for the problem* (1)*, f be a forcing. Then* (1) *rewritten in the Fourier basis become*

$$\frac{da_k}{dt} = -i\frac{k}{2} \sum_{k_1\in\mathbb{Z}} a_{k_1} \cdot a_{k-k_1} - \nu k^2 a_k + f_k, \quad k \in \mathbb{Z}, \tag{5a}$$

$$a_k(0) = \frac{1}{2\pi} \int_0^{2\pi} u_0(x) e^{-ikx}\, dx, \quad k \in \mathbb{Z}, \tag{5b}$$

$$f_k = \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-ikx}\, dx, \quad k \in \mathbb{Z}. \tag{5c}$$

Proof. (5b) and (5c) is immediate consequence of (1c) , (1d) and (2). Our goal is to rewrite (1a) in the Fourier basis. Using (3) the nonlinear term reads as follows

$$u \cdot \frac{du}{dx} = \sum_{k\in\mathbb{Z}} a_{k_1} e^{ik_1 x} \cdot \sum_{k_2\in\mathbb{Z}} ik_2 a_{k_2} e^{ik_2 x} = i \sum_{k_1,k_2} e^{i(k_1+k_2)x} k_2 a_{k_1} \cdot a_{k_2}.$$

We set $k = k_1 + k_2$

$$i \sum_{k_1,k_2} e^{i(k_1+k_2)x} k_2 a_{k_1} \cdot a_{k_2} = i \sum_{k\in\mathbb{Z}} \left(\sum_{k_1\in\mathbb{Z}} (k-k_1) a_{k_1} \cdot a_{k-k_1} \right) \cdot e^{ikx}.$$

(1a) in the Fourier basis reads as follows

$$\sum_{k\in\mathbb{Z}} \frac{da_k}{dt} e^{ikx} = -i \sum_{k\in\mathbb{Z}} \left(\sum_{k_1\in\mathbb{Z}} (k-k_1) a_{k_1} \cdot a_{k-k_1} \right) e^{ikx} - \nu \sum_{k\in\mathbb{Z}} k^2 a_k e^{ikx} + \sum_{k\in\mathbb{Z}} f_k e^{ikx}.$$

Having in mind fact that the Fourier basis is orthogonal and comparing the coefficients finally we obtain the infinite ladder of ODEs

$$\frac{da_k}{dt} = -i \sum_{k_1\in\mathbb{Z}} (k-k_1) a_{k_1} \cdot a_{k-k_1} - \nu k^2 a_k + f_k, \quad k \in \mathbb{Z}.$$

Using Lemma 2.3 we we obtain the final form

$$\frac{da_k}{dt} = -i\frac{k}{2} \sum_{k_1\in\mathbb{Z}} a_{k_1} \cdot a_{k-k_1} - \nu k^2 a_k + f_k, \quad k \in \mathbb{Z}. \quad \blacksquare$$

3 Estimates for the nonlinear part used in the integration algorithm

The foundation of our algorithm calculating the rigorous enclosures for the solutions of (1a) is the spectral method, see [CHQZ]. This kind of algorithms are applied to the problem of calculating approximate solutions of PDEs with an initial value and periodic boundary conditions. In our case, as we are interested in rigorous results only, we have to perform significant modifications. In particular we consider the whole convolution in (5a) not only a truncation.

In this section we derive estimates for nonlinear term of the form $c(u^2)_x$ that we use in our algorithms. This form of nonlinear part is exhibited by a class of dPDEs that include the Burgers equation, the Navier-Stokes equations and the Kuramoto-Shivasinsky equation.

Notation We use following notation: $\cdot$ is multiplication in $\mathbb{C}$, $e(k) = 1$ for k odd and 0 for k even. $|\cdot|$ is the Euclidean norm, sometimes we will use this symbol in the context of interval sets ($[a] = \Pi_{k=1}^{n}[a_k^-, a_k^+]$) $|[a]| := \operatorname{mag}(a) = \max\{|x| \colon x \in a\}$ and as the absolute value of an integer, we do not distinct those notations and it will always be clear from context what we mean. We denote by $\mathrm{B}(r) := \overline{B(0,r)}$ a zero-centered $\mathbb{R}^2$ ball of radius r and by $\mathrm{Sq}(r)$ a smallest zero centered square containing $\mathrm{B}(r)$. Moreover, let $\{a_k\}_{k\in\mathbb{Z}} \in H$ we define

$$\lambda_k := -\nu k^2, \quad S_k\left(\{a_k\}_{k\in\mathbb{Z}}\right) := \sum_{k_1\in\mathbb{Z}} a_k \cdot a_{k-k_1}, \quad N_k\left(\{a_k\}_{k\in\mathbb{Z}}\right) := -\frac{ik}{2} \cdot S_k\left(\{a_k\}_{k\in\mathbb{Z}}\right).$$

Definition 3.1. Let $W \subset H$, W convex. We call W the polynomial bound if there exists numbers $M > 0$, $C > 0$, $s \geq 0$ such that

$$|W_i| \leq \frac{C}{|i|^s}, \; |i| > M. \tag{6}$$

To denote the polynomial bound we use the quadruple (W, M, C, s).

The goal of this section is given a polynomial bound

$$(W \oplus T, M, C, s), \quad W \oplus T \subset H,$$

provide a polynomial bound

$$\left(\widetilde{N}, M, C_{\widetilde{N}}, s_{\widetilde{N}}\right), \quad \widetilde{N} \subset H$$

such that

$$N_k(W \oplus T) = \{N_k\left(\{a_k\}_{k\in\mathbb{Z}}\right) \mid \{a_k\}_{k\in\mathbb{Z}} \in W \oplus T\} \subset \widetilde{N}_k, \quad k \in \mathbb{Z}.$$

We consider two sub-cases

Case 1 $0 < k \leq M$. We calculate $\widetilde{N}_k \subset \mathbb{R}^2$ such that $N_k(W \oplus T) \subset \widetilde{N}_k$ explicitly with relatively high accuracy. All estimates are provided for N_k where $k > 0$, estimates for N_{-k} are obtained by imposing the symmetry $\widetilde{N}_{-k} = \overline{\widetilde{N}_k}$.

Case 2 $k > M$. We calculate numbers $C_{\widetilde{N}}$ and $s_{\widetilde{N}}$ such that $|\widetilde{N}_k| \leq C_{\widetilde{N}}/k^{s_{\widetilde{N}}}$.

3.1 Case 1, $\widetilde{N}_k$ for $0 < k \leq M$

In this case we are interested in providing an explicit bounds $\widetilde{N}_k \subset \mathbb{R}^2$ such that $N_k(W \oplus T) \subset \widetilde{N}_k$. To achieve this goal, we estimate $S_k(W \oplus T)$ using an explicit formulas. Note that $S_k \subset \mathbb{R}^2$, we provide estimations for both components simultaneously and we do not expand the multiplications for better clarification.

3.1.1 $0 < k \leq m$

$$S_k(W \oplus T) \subset \sum_{k-M \leq k_1 \leq M} a_{k_1} \cdot a_{k-k_1} + \sum_{M < k_1 \leq M+k} a_{k-k_1} \cdot \mathrm{Sq}\left(\frac{C}{|k_1|^s}\right)$$
$$+ \sum_{-M \leq k_1 < k-M} a_{k_1} \cdot \mathrm{Sq}\left(\frac{C}{|k-k_1|^s}\right) + \sum_{k_1 < -M \vee k_1 > M+k} \mathrm{Sq}\left(\frac{2C^2}{|k_1|^s |k-k_1|^s}\right).$$

This sum is symmetric (with respect to $k_1 \to k - k_1$) and can be rewritten in simpler form using this symmetry

$$S_k(W \oplus T) \subset e(k) a_{\frac{k}{2}}^2 + 2 \sum_{\frac{k}{2} < k_1 \leq M} a_{k_1} \cdot a_{k-k_1} + 2 \sum_{M < k_1 \leq M+k} \mathrm{Sq}\left(\frac{C}{k_1^s}\right) \cdot a_{k-k_1}$$
$$+ 2 \sum_{k_1 > M+k} \mathrm{Sq}\left(\frac{2C^2}{|k_1|^s |k-k_1|^s}\right).$$

3.1.2 $m < k \leq M$

Since in this case k may be large, we perform additional estimations for the terms involving the far tail.

$$S_k(W \oplus T) \subset e(k) a_{\frac{k}{2}}^2 + 2 \sum_{k-M \leq k_1 < \frac{k}{2}} a_{k_1} \cdot a_{k-k_1}$$
$$+ 2 \sum_{-M \leq k_1 < k-M} a_{k_1} \cdot \mathrm{Sq}\left(\frac{C}{|k-k_1|^s}\right) + 2 \sum_{k_1 < -M} \mathrm{Sq}\left(\frac{2C^2}{|k_1|^s |k-k_1|^s}\right).$$

Let us focus on the term $2\sum_{-M \leq k_1 < k-M} a_{k_1} \cdot \mathrm{Sq}\left(\frac{C}{|k-k_1|^s}\right)$. When k is close to M it may contain a large number of terms $\approx M$, regarding the fact that calculating products is expensive in computational point of view, we estimate this term

$$\sum_{-M\le k_1<k-M} a_{k_1}\cdot \operatorname{Sq}\left(\frac{C}{|k-k_1|^s}\right) \subset \operatorname{Sq}\left(\frac{C}{(M+1)^s}\right)\cdot \operatorname{Sq}\left(\sum_{-M\le k_1<k-M} |a_{k_1}|\right)$$
$$\subset 2\operatorname{Sq}\left(\frac{C}{(M+1)^s}\cdot \sum_{-M\le k_1<0} |a_{k_1}|\right).$$

And as a result we receive

$$S_k(W\oplus T) \subset 2\sum_{k-M\le k_1<\frac{k}{2}} a_{k_1}\cdot a_{k-k_1} + e(k)a^2_{\frac{k}{2}}$$
$$+ 4\operatorname{Sq}\left(\frac{C}{(M+1)^s}\sum_{-M\le k_1<0}|a_{k_1}|\right) + 2\sum_{k_1<-M}\operatorname{Sq}\left(\frac{2C^2}{|k_1|^s|k-k_1|^s}\right).$$

There appears an infinite sum. We denote it by

$$\operatorname{IS}(k) := \sum_{k_1<-M}\frac{C^2}{|k_1|^s|k-k_1|^s} = \sum_{k_1>M+k}\frac{C^2}{|k_1|^s|k-k_1|^s}.$$

Lemma 3.2.

$$\operatorname{IS}(k) < \frac{C^2}{(2s-1)\left[M(k+M)\right]^{s-\frac{1}{2}}}.$$

Proof.

$$\operatorname{IS}(k) \le C^2\sqrt{\sum_{k_1<-M}\frac{1}{|k_1|^{2s}}}\sqrt{\sum_{k_1<-M}\frac{1}{|k-k_1|^{2s}}},$$

due to Cauchy-Schwarz inequality. We estimate separately both square rooted sums, this estimates will be used throughout this section

$$\sum_{k_1<-M}\frac{1}{|k_1|^{2s}} < \int_M^\infty \frac{1}{r^{2s}}\,dr = \left[-\frac{1}{(2s-1)r^{2s-1}}\right]_M^\infty = \frac{1}{(2s-1)M^{2s-1}}$$

and analogically

$$\sum_{k_1<-M}\frac{1}{|k-k_1|^{2s}} \le \frac{1}{(2s-1)(k+M)^{2s-1}},$$

eventually

$$\sum_{k_1<-M}\frac{C^2}{|k_1|^s|k-k_1|^s} < \frac{C^2}{(2s-1)\left[M(k+M)\right]^{s-\frac{1}{2}}}. \quad \blacksquare$$

3.2 Case 2, $\widetilde{N}_k$ for $k > M$

In this case we are interested in computing numbers $C_{\widetilde{N}}$ and $s_{\widetilde{N}}$ such that $|\widetilde{N}_k| \leq C_{\widetilde{N}}/k^{s_{\widetilde{N}}}$ for $k > M$, we assume that $s > 1$ to have all necessary series convergent.

Case 2.1 $k > 2M$

$$S_k(W \oplus T) \subset \sum_{k-M \leq k_1 \leq M+k} \mathrm{B}\left(\frac{C}{|k_1|^s}\right)|a_{k-k_1}| + \sum_{-M \leq k_1 \leq M} |a_{k_1}| \,\mathrm{B}\left(\frac{C}{|k-k_1|^s}\right)$$
$$+ \sum_{k_1 < -M \,\vee\, k_1 > M+k \,\vee\, M < k_1 < k-M} \mathrm{B}\left(\frac{C}{|k_1|^s}\right) \cdot \mathrm{B}\left(\frac{C}{|k-k_1|^s}\right).$$

Rewritten using symmetry with respect to $k_1 \to k - k_1$ reads

$$S_k(W \oplus T) \subset 2 \sum_{M < k_1 < \frac{k}{2}} \mathrm{B}\left(\frac{C}{|k_1|^s}\right) \cdot \mathrm{B}\left(\frac{C}{|k-k_1|^s}\right) + e(k)\,\mathrm{B}\left(\frac{2^s C}{|k|^s}\right)^2$$
$$+ 2 \sum_{-M \leq k_1 \leq M} |a_{k_1}| \,\mathrm{B}\left(\frac{C}{|k-k_1|^s}\right) + 2 \sum_{k_1 < -M} \mathrm{B}\left(\frac{C}{|k_1|^s}\right) \cdot \mathrm{B}\left(\frac{C}{|k-k_1|^s}\right). \quad (7)$$

Case 2.2 $M < k \leq 2M$

$$S_k(W \oplus T) \subset \sum_{M < k_1 \leq M+k} \mathrm{B}\left(\frac{C}{|k_1|^s}\right)|a_{k-k_1}| + \sum_{-M \leq k_1 < k-M} |a_{k_1}| \,\mathrm{B}\left(\frac{C}{|k-k_1|^s}\right)$$
$$+ \sum_{k_1 < -M \,\vee\, k_1 > M+k} \mathrm{B}\left(\frac{C}{|k_1|^s}\right) \cdot \mathrm{B}\left(\frac{C}{|k-k_1|^s}\right) + \sum_{k-M \leq k_1 \leq M} a_{k_1} \cdot a_{k-k_1}.$$

We use the symmetry (with respect to $k_1 \to k - k_1$) like in Case 2.1

$$S_k(W \oplus T) \subset 2 \sum_{k-M \leq k_1 < \frac{k}{2}} a_{k_1} \cdot a_{k-k_1} + e(k) a_{\frac{k}{2}}^2$$
$$+ 2 \sum_{-M \leq k_1 < k-M} |a_{k_1}| \,\mathrm{B}\left(\frac{C}{|k-k_1|^s}\right) + 2 \sum_{k_1 < -M} \mathrm{B}\left(\frac{C}{|k_1|^s}\right) \cdot \mathrm{B}\left(\frac{C}{|k-k_1|^s}\right). \quad (8)$$

the infinite sum We denote the infinite sum, which appear in estimations above by

$$\mathrm{IS}_k := \sum_{k_1 < -M} \mathrm{B}\left(\frac{C}{|k_1|^s}\right) \cdot \mathrm{B}\left(\frac{C}{|k-k_1|^s}\right), \text{ for } k > M$$

and estimate it separately

$$|\,\mathrm{IS}_k\,| = \left| \sum_{k_1<-M} \mathrm{B}\left(\frac{C}{|k_1|^s}\right) \cdot \mathrm{B}\left(\frac{C}{|k-k_1|^s}\right) \right| \le \frac{1}{|k|^s} \sum_{k_1<-M} \frac{C^2}{|k_1|^s|1-\frac{k_1}{k}|^s}$$
$$\le \frac{C^2}{|k|^s(s-1)M^{s-1}}.$$

Case 2.1 $k > 2M$, by $FS1_k$ we denote the finite part of the sum (7) and estimate it

$$|FS1_k| \le 2 \sum_{M<k_1<\frac{k}{2}} \left| \mathrm{B}\left(\frac{C}{|k_1|^s}\right) \cdot \mathrm{B}\left(\frac{C}{|k-k_1|^s}\right) \right|$$
$$+ e(k) \left| \mathrm{B}\left(\frac{2^s C}{|k|^s}\right)^2 \right| + 2 \sum_{-M\le k_1\le M} \left| a_{k_1} \cdot \mathrm{B}\left(\frac{C}{|k-k_1|^s}\right) \right|$$
$$\le \frac{2C}{|k|^s} \left(\sum_{M<k_1<\frac{k}{2}} \frac{C}{|k_1|^s|1-\frac{k_1}{k}|^s} + \frac{\frac{1}{2}C4^s}{|k|^s} + \sum_{-M\le k_1\le M} |a_{k_1}| \frac{1}{|1-\frac{k_1}{k}|^s} \right)$$
$$\le \frac{2C}{|k|^s} \left(\frac{C2^s}{(s-1)M^{s-1}} + \frac{\frac{1}{2}C4^s}{(2M+1)^s} + 2^s \sum_{-M\le k_1\le M} |a_{k_1}| \right).$$

Finally, for $k > 2M$

$$|S_k| \le 2|IS_k| + |FS1_k|.$$

Case 2.2 $M < k \le 2M$, by $FS2_k$ we denote the finite part of the sum (8) and estimate it

$$|FS2_k| \le 2 \sum_{-M\le k_1<k-M} \left| a_{k_1} \cdot \mathrm{B}\left(\frac{C}{|k-k_1|^s}\right) \right|$$
$$+ \left| 2 \sum_{k-M\le k_1<\frac{k}{2}} a_{k_1} \cdot a_{k-k_1} + e(k) a^2_{\frac{k}{2}} \right| \le \frac{2}{|k|^s} \sum_{-M\le k_1<k-M} |a_{k_1}| C2^s$$
$$+ \left| 2 \sum_{k-M\le k_1<\frac{k}{2}} a_{k_1} \cdot a_{k-k_1} + e(k) a^2_{\frac{k}{2}} \right| \le \frac{2^{s+1}C}{|k|^s} \sum_{-M\le k_1<k-M} |a_{k_1}|$$
$$+ \left| 2 \sum_{k-M\le k_1<\frac{k}{2}} a_{k_1} \cdot a_{k-k_1} + e(k) a^2_{\frac{k}{2}} \right|.$$

Finally, for $M < k \le 2M$

$$|S_k| \le 2|IS_k| + |FS2_k|.$$

final estimates

$$D_\infty := 2\frac{C^2}{(s-1)M^{s-1}},$$

$$D1 := 2^{s+1}C\left(\frac{C}{(s-1)M^{s-1}} + \frac{C2^{s-1}}{(2M+1)^s} + \sum_{-M\le k_1\le M}|a_{k_1}|\right),$$

$$D2 := 2\left(C2^s\sum_{-M\le k_1<M}|a_{k_1}| + |k|^s\left|\sum_{k-M\le k_1<\frac{k}{2}} a_{k_1}\cdot a_{k-k_1} + \frac{1}{2}e(k)a^2_{\frac{k}{2}}\right|\right),$$

$$D_F := \max\{D1,\ D2\}.$$

The following holds

$$|FS1_k| \le \frac{D1}{|k|^s},\quad |FS2_k| \le \frac{D2}{|k|^s},\quad 2\,|IS_k| \le \frac{D_\infty}{|k|^s}.$$

Therefore, estimate for $|S_k|$, $k > M$ reads

$$|S_k| \le \frac{D_F + D_\infty}{|k|^s},$$

finally

$$\frac{|k|}{2}|S_k| = |N_k| \le \frac{\frac{1}{2}\left(D_F + D_\infty\right)}{|k|^{s-1}} =: C_{\widetilde{N}}/k^{s_{\widetilde{N}}}.$$

4 Algorithm for inverting matrices

In order to find a rigorous inverses of matrices we have used *the Krawczyk operator*

Definition 4.1. Let $F(x) = 0$ be a system of equations. The Krawczyk operator for finding zeros of $F(x)$ takes the form

$$K([x], m, F) = m - C\cdot F(m) + (Id - C\cdot A)\,[r],$$

where $[x]$ is an interval set within which we are looking for zeros of $F(x)$, $[x] = m + [r]$ and usual choice of $C \approx dF(m)^{-1}$.

Theorem 4.2. [N] *If $x^* \in [x]$ and $F(x^*) = 0$, then $x^* \in K\left(m, [x], F\right)$.*

```
Input: single valued matrix A, single valued matrix C ≈ A^{-1}
Output: interval matrix X
X := C + [-1,1]// add to each element of C the interval [-1,1];
while !found do
    found := true;
    for step := 0,...,4 do
        foreach column (X)_i of X do
            x_i := (X)_i // (X)_i is the i-th column of the matrix X;
            m := mid(x_i);
            [r] := x_i - m;
            (K)_i := m - C·(A·m - (Id)_i) + (Id - C·A)[r];
        end
        if X ∩ K = ∅ then
            X := X + [-1,1];
            found := false;
            break;
        end
        X := X ∩ K;
    end
end
return X;
```

Algorithm 1: Rigorous inverse matrix

Lemma 4.3. *Let A be a given single valued matrix. The matrix X calculated using the Algorithm 1 contains the matrix A^{-1} such that $A \cdot A^{-1} = Id$.*

Proof. $A^{-1} \in X$ follows immediately from the fact that linear equations $A \cdot (X)_i - (Id)_i = 0$ are independent and the fundamental property of the Krawczyk operator, Theorem 4.2. ∎

References

[Package] Program package, http://www.ii.uj.edu.pl/~cyranka/Burgers.

[C] J. Cyranka, *Existence of globally attracting fixed points of viscous Burgers equation with constant forcing. A computer assisted proof*, submitted 2011.

[CHQZ] C. Canuto, M.Y. Hussaini, A. Quarteroni, T.A. Zang, *Spectral Methods. Fundamentals in Single Domains.* Springer, 2006.

[N] A. Neumeier, *Interval methods for system of equations.* Cambridge University Press, 1990.

[Wh] G. B. Whitham, *Linear and Nonlinear Waves.* John Wiley & Sons, 1975.