SCRITICAL PAIRS AND SUNIFICATION

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SOME NOTATION

Given S-sorted signature Σ, variable sets X, Y:

$T_\Sigma(X)$ is free Σ-algebra generated by X, i.e., all Σ-terms with vars from X;
substitution is $\theta\colon X \to T_\Sigma(Y)$;
n-**ary context** is $\gamma \in T_\Sigma(X \cup Z)$, $Z = \{z_1, ..., z_n\}$, each $z_1, ..., z_n \notin X$ occurs once in γ.
E.g. $\gamma[z]$ unary context,
$\gamma[z_1, z_2]$ or $\gamma[z_1][z_2]$ or $\gamma[\cdot][\cdot]$ binary context.

$T_{\Sigma,B}(X)$ is free (Σ, B)-algebra generated by X.
class substitution is $\theta\colon X \to T_{\Sigma,B}(Y)$.
n-**ary class context** is $c \in T_{\Sigma,B}(X \cup Z)$, $Z = \{z_1, ..., z_n\}$, $z_1, ..., z_n \notin X$.
E.g. $z + a + z$ unary context modulo associativity of $+$, with a a constant.

REWRITING MODULO EQUATIONS

Many sorted term rewriting system (TRS) is a set of Σ-rewrite rules, i.e., pairs of Σ-terms, $t \to t'$ with $var(t') \subseteq var(t)$.

B set of Σ-equations, A Σ-TRS, then (Σ, A, B) is **many sorted term rewriting system modulo equations (MTRS)**.

For $t_1, t_2 \in T_\Sigma(X)$, define $t_1 \Rightarrow_{A/B} t_2$ (t_1 **rewrites to** t_2 **with** A **modulo** B) iff $\exists\ t'_1, t'_2$ such that $t_1 \simeq_B t'_1$, $t'_1 \Rightarrow_A t'_2$, $t_2 \simeq_B t'_2$

Locally confluent MTRS iff $t \Rightarrow_{A/B} t_1$ and $t \Rightarrow_{A/B} t_2$ imply exists t' such that $t_1 \overset{*}{\Rightarrow}_{A/B} t'$ and $t_2 \overset{*}{\Rightarrow}_{A/B} t'$.

OVERLAP MODULO EQUATIONS

Subterm modulo B of $t \in T_{\Sigma}(X)$ is $t' \in T_{\Sigma}(X)$ such that $t \simeq_B \gamma[t']$ for some γ; write $t \ll_B t'$.

Two terms **overlap modulo** B iff one unifies modulo B with a subterm modulo B of other, and subterm is not just a variable.

E.g. ℓ_1, ℓ_2 overlap modulo B iff
$\exists\ \theta$ substitution, $\gamma[t_0]$ term such that
$t_0 \notin var(\ell_1)$, $\ell_1 \simeq_B \gamma[t_0]$, $\theta(t_0) \simeq_B \theta(\ell_2)$.

θ is the (**modulo** B) **overlap** of ℓ_1, ℓ_2 at subterm (modulo B) t_0 of ℓ_1.

Given modulo B overlaps θ, θ' of t, t' at subterms modulo B t_0, t'_0 of t such that $t \simeq_B \gamma[t_0] \simeq_B \gamma'[t'_0]$, $\gamma \simeq_B \gamma'$ and $t_0 \simeq_B t'_0$, then θ **subsumes** θ' iff exists substitution ρ such that $\theta' \simeq_B \theta; \rho$.

Proposition: If equivalence classes modulo B are finite then finite complete sets of unifiers modulo B exist iff finite complete sets of overlaps modulo B exist.

MOTIVATION

Given $\ell_1 \to r_1, \ell_2 \to r_2$ rules that overlap, $\langle \theta r_1, \theta\gamma[\theta r_2] \rangle$ is **critical pair of the overlap**.

Theorem [Knuth-Bendix]: A TRS is locally confluent iff all its critical pairs are convergent.

The "if" part is false for MTRS:

Ex: $\Sigma = \{+, a, b\}$, B is associativity for $+$,
$A = \{a + b \to b,\ b + a \to a\}$
No critical pairs since A non-overlapping.
But (Σ, A, B) not locally confluent:
$a + b + a \overset{*}{\Rightarrow}_{A/B} a, \quad a + b + a \Rightarrow_{A/B} a + a$

Solution: Generalize to scritical pairs.
Provide sufficient condition for overlap modulo B to work for MTRS.

SUNIFIERS

Sunifier (**superunifier**) of $\ell_1, \ell_2 \in T_\Sigma(X)$
is $\diamondsuit = (\theta, u_1, u_2, t)$
where $\theta\colon X \to T_\Sigma(Y)$ substitution,
$t \in T_\Sigma(Y)$, $u_1, u_2 \in T_\Sigma(Y \cup \{z\})$ contexts
such that $u_1[\theta(\ell_1)] \simeq_B u_2[\theta(\ell_2)] \simeq_B t$.

$\langle u_1[\theta r_1], u_2[\theta r_2]\rangle$ is associated **scritical** (or **supercritical**) **pair**

If $u_1 = u_2 = z$ then θ unifier.
If θ unifier modulo B of t_1, t_2 then $(\theta, z, z, \theta t_1)$ is **corresponding** sunifier.

If $u_1 = z$ or $u_2 = z$ then θ overlap. If θ overlap modulo B of t_1, t_2 at $t_0 \ll_B t_2$ with $t_2 \simeq_B \gamma[t_0]$ then $(\theta, z, \gamma, \theta t_2)$ is **corresponding** sunifier.

Ex (cont): Sunifiers of $a + x$, $b + y$:
$((x \mapsto b,\ y \mapsto y), z + y, a + z, a + b + y)$
$((x \mapsto b,\ y \mapsto b), z+b+a, a+z+a, a+b+b+a)$

SUBSUMPTION

$\diamondsuit = (\theta, u_1, u_2, t)$ **subsumes** $\diamondsuit' = (\theta', u_1', u_2', t')$, sunifiers of t_1, t_2, iff $\exists \rho$ substitution, u context such that $\theta' \simeq_B \theta; \rho$ and $u_i'[\cdot] \simeq_B u[\rho(u_i[\cdot])]$ for $i = 1, 2$.
If $u \neq z$ then $\diamondsuit$ **strictly** subsumes $\diamondsuit$.

Minimal sunifier is one that is minimal with respect to strict subsumption.

Ex (cont):
$((x \mapsto b,\ y \mapsto y), z + y, a + z, a + b + y)$ subsumes $((x \mapsto b,\ y \mapsto b), z + b + a, a + z + a, a + b + b + a)$ with $\rho = (y \mapsto b), u = z + a$.

Subsumption is reflexive, transitive, not total:

Ex (cont): $((), z + a, a + z, a + b + a)$ and $((), b + z, z + b, b + a + b)$ both sunifiers of $a + b, b + a$ but neither subsumes the other.

Proposition: $\diamondsuit$ subsumes $\diamondsuit'$ and scritical pair of $\diamondsuit$ converges implies scritical pair of $\diamondsuit'$ converges.

SEPARATED SUNIFIERS

$t_1, t_2 \in T_\Sigma(X)$ **separated** (modulo B) in $t \in T_\Sigma(X)$ iff $t_1, t_2 \ll_B t$ and exists binary context γ such that $t \simeq_B \gamma[t_1, t_2]$.

Sunifier (θ, u_1, u_2, t) of t_1, t_2 **separated** iff $\theta(t_1), \theta(t_2)$ separated in t.

Ex (cont): No sunifier so far is separated, but $((), z+b+a, a+b+z, a+b+b+a)$ is separated sunifier of $a+b, b+a$.

Proposition: If B consists of A/C laws then scritical pairs of minimal separated sunifiers converge.

There are B such that separated sunifiers generate non-convergent scritical pairs.

Ex: $\Sigma = \{h, f, g, i, a, b, c\}$,
$B = \{f(i(a)) = g(i(b)) = h(i(a), i(b))\}$,
$A = \{i(a) \to a, i(b) \to c\}$.
All sunifiers of $i(a)$ and $i(b)$ are separated, but MTRS is not confluent modulo B.

COMPLETE SETS OF SUNIFIERS

Set I of [unseparated] sunifiers of t_1, t_2 **complete** iff every [unseparated] sunifier of t_1, t_2 subsumed by one in I.

Complete set of [unseparated] sunifiers of t_1, t_2 **minimal** iff no proper subset also complete.

Proposition: Complete sets of [unseparated] sunifiers always exist and are at most countable. The unseparated ones might be empty.

Conclusion: if complete (preferably minimal) set of sunifiers exists, suffices to use only the scritical pairs corresponding to these sunifiers.

In fact use "almost complete" sets of sunifiers:

Sufficient set of [unseparated] sunifiers iff every [unseparated] sunifier is either subsumed by one in the set, or else its scritical pair is convergent.

KNUTH-BENDIX FOR MTRS

Theorem: An MTRS is locally confluent iff all its scritical pairs converge.

Similar Peterson-Stickel result involves unifiers of all "variable extensions of rules".
Always gives infinity of pairs to check.

Scritical pair sets can be finite.

SUNIFIERS MODULO C, AC, AC1

Complete sets of sunifiers modulo associativity do not always exist:

Ex: Substitution $\theta_n(x) = a + na$, $\theta_n(y) = na$.
$(\theta_n, z, z, a + na + a + na)$ sunifies
$x + x$ and $a + y + y + a$.
Infinite set of sunifiers cannot be subsumed by any finite set of sunifiers.

Theorem: Finite minimal sufficient sets of sunifiers modulo C, AC and AC1 always exist.

Finite sufficient AC, AC1 sunifier sets for ℓ_1, ℓ_2:

sunifiers corresponding to finite complete sets of unifiers for:

(ℓ_1, ℓ_2); $(\ell_1, \ell_2 + y_2)$;
$(\ell_1 + y_1, \ell_2)$; $(\ell_1 + y_1, \ell_2 + y_2)$

where y_1, y_2 not in $var(\ell_1) \cup var(\ell_2)$.

SUFFICIENT CONDITION

(C) *For all* Σ*-terms* t, t', u *such that* $t \simeq_B t'$ *and* $t = \gamma[u]$ *for some context* $\gamma[\cdot]$*, there exist a context* $\gamma'[\cdot]$ *and* Σ*-term* u' *such that* $\gamma \simeq_B \gamma'$*,* $u \simeq_B u'$ *and* $t' = \gamma'[u']$*.*

If B satisfies (C) then
overlapping & critical pairs
can replace
sunifiers & scritical pairs.

(C) satisfied by (n-ary) commutative operators, but neither by A nor AC operators.

Proposition: If (C) holds for B, then sunifiers modulo B are either overlaps or else have convergent scritical pairs.

Theorem: An MTRS that satisfies (C) is locally confluent iff all its critical pairs converge.

Conclusion: algorithm to decide confluence modulo commutativity based on algorithm to compute complete sets of overlaps (in fact complete sets of unifiers).

Same result using sunifiers follows from:

Proposition: If B satisfies (C) and finite complete sets of overlaps modulo B exist, then finite sufficient sets of sunifiers also exist, and the resulting scritical pairs are the critical pairs of the overlaps.

CATEGORY THEORY GENERALIZATION

Applies to free objects, e.g., term algebras with additional structure, such as many sorts, ordered sorts, equationally defined subsorts, or continuity.

Can define sunifiers, scritical pairs, separation, subsumption, and get main theorem.

Probably can do much more ...

RELATED WORK

Knuth-Bendix [1970] approach fails to handle permutative equations like commutativity, since equations oriented as rewrite rules.

Huet [1980] assumes left linear rules and balanced equations. Confluence not that of $\Rightarrow_{A/B}$. Critical pairs obtained by overlapping rules with one another and with equations in B.

Peterson and Stickel [1981] restrict to balanced, linear equations. "Variable extension" notion resembles sunifier notion. Solution to infinitely many variable extensions is restricting to reductions compatible with the equations.

Jouannaud and Kirchner [1986] generalize compatibility of Peterson and Stickel. Critical pair algorithm assumes finite complete unification and finite equivalence classes.

All treat unsorted terms.

CONCLUSION

Our approach to rewriting modulo equations and confluence modulo equations attempts to:
– lend greater unity and generality to the area
– provide general basis for modular algorithms

Sunifier notion is simple and natural.

Suits well the C, AC and AC1 cases.

Expect to treat more cases, e.g. linear A, and extend category theoretic approach.