Chapter 5

INFINITE SERIES

Let's Get Acquainted With The Math Tables In The Appendix

By now, you have had much practice with differential and integral calculus. You have surely noticed that the techniques of integration were harder to master than the techniques of differentiation. Don't feel bad about this situation. It's the same for everyone who studies calculus. If you don't believe that this is the case, look at the math tables in the appendix of this book. Notice how much more space is devoted to tables of integrals than to tables of derivatives! In fact, now is a good time to take a few minutes and browse through the math tables in the appendix. In particular, look at the section labled SERIES and the TABLE OF INTEGRALS. Look in the TABLE OF INTEGRALS under the subsection TRANSCENDENTAL FUNCTIONS and find $\int \frac{\sin(x)}{x} dx$.

What you should have found is the statement

$$\int \frac{\sin(x)}{x} = x - \frac{x^3}{3 \cdot 3!} + \frac{x^5}{5 \cdot 5!} - \frac{x^7}{7 \cdot 7!} + \frac{x^9}{9 \cdot 9!} \cdots$$

This strange expression for $\int \frac{\sin(x)}{x}$ is called an "infinite series expansion of the integral of $\frac{\sin(x)}{x}$." For a moment, let's ignore the question of how this infinite series expansion was derived and just try to understand what it means. Notice first of all that the series is a sum of powers of x. In this case, all powers are odd. The terms alternate in sign. For any odd number n, the power x^n is divided by $n \cdot n!$. The number $n \cdot n!$ is obtained by multiplying the integer n times the integer n!. The integer n! is the product $(n)(n - 1)(n - 2) \cdots (2)(1)$. These numbers n! grow very rapidly as n increases. The series ends with three little dots ". . ." which is meant to imply that anyone should be able to figure out the general term from the terms already given. You should be aware, however, that sometimes series are terminated with . . . without anybody really knowing the general rule! Students, in particular, love to do this. Sometimes, the general rule for forming the series is quite complicated. Look in the math tables in the section labeled SERIES at the infinite series for tan(x) and ctn(x) where the general term is expressed as a function of the Bernoulli numbers.

An Infinite Series Specifies Better And Better Polynomial Approximations

We still haven't discussed what the infinite series for $\int \frac{\sin(x)}{x}$ really means. Recall that in the beginning of CHAPTER 4, INTEGRATION, we discussed the idea of the signed area function. FIGURE 4.6 is particularly relevant here. There we saw that given a function f(x), we could always produce an integral F(x), the "signed area function," by graphical methods. In particular, we could apply this technique to compute $F(x) = \int \frac{\sin(x)}{x}$. Imagine that we have done this with great accuracy and we have graphed F(x), as was done in FIGURE 4.6. Assume that we have taken the base point for the signed area function to be $x = 0$, so $F(0) = 0$. Now look once again at the infinite series expansion for $\int \frac{\sin(x)}{x}$. Notice, if we throw away all terms in the infinite series involving powers of x larger than k, we obtain a polynomial. For $k = 1, 3, 5$, and 7, for example, we would obtain the polynomials x, $x - \frac{x^3}{3 \cdot 3!}$, $x - \frac{x^3}{3 \cdot 3!} + \frac{x^5}{5 \cdot 5!}$, and $x - \frac{x^3}{3 \cdot 3!} + \frac{x^5}{5 \cdot 5!} - \frac{x^7}{7 \cdot 7!}$. Let's call these polynomials $p_1(x)$, $p_3(x)$, $p_5(x)$, and $p_7(x)$. If we were to compare the graphs of these polynomials with the graph of F(x), starting at $x = 0$ and moving away, we would see that $p_1(x)$ looks almost identically like the graph of F(x) for very small values of x. We would see that the graph of $p_3(x)$ looks almost identically like the graph of F(x) for an even wider interval of values about $x = 0$.

Similarly, $p_5(x)$ and $p_7(x)$ are even better approximations to F(x) over a wider range of values of F(x) centered at x = 0. This is the fundamental idea of what the statement of equality between any function F(x) and its infinite series (sometimes called its ''power series'') is to mean. The infinite series is simply a concise way of specifying a sequence of polynomials that are better and better approximations to the function about some value of x (x = 0 in our example).

Some Simple-Looking Functions Are Very Nasty To Integrate

But wait a minute! Why did we bother to express the integral $\int \frac{\sin(x)}{x}$ as an infinite series? We didn't do that sort of thing in CHAPTER 4. We wrote $\int \sin(x)dx = -\cos(x)$, for example. We didn't express $\int \sin(x)dx$ as an infinite series. Why don't we write $\int \frac{\sin(x)}{x} = -\frac{\cos(x)}{x}$ or, if that's not quite right, something like that? The answer is that, try as they may, no one has been able to find a reasonable-looking expression for $\int \frac{\sin(x)}{x}dx$ in terms of the familiar functions used to express integrals in CHAPTER 4. Mathematicians express this fact by saying ''the integral $\int \frac{\sin(x)}{x}dx$ cannot be expressed in terms of elementary functions.'' This may sound a little condescending to you, as the functions we have been using thus far in this book may not seem so ''elementary,'' but this is what is said. Thus, the way we deal with ''nasty to integrate functions'' like $\frac{\sin(x)}{x}$ is to approximate them with polynomials. Polynomials are nice functions. The bad news about this is that any given polynomial is only a good approximation for certain restricted values of the variable x. Usually, these values are specified as lying in some interval about a fixed value of x. We speak of these polynomials as ''local approximations'' to our nasty function. When we do our calculus tricks on these local approximations, such as find areas, volumes, line integrals, etc., we have to be careful. We must always remember that we are doing ''local calculus.'' This is not to say that you should be paralyzed with fear at the thought of making a mistake with local calculus. By far the best strategy is to plow ahead and be daring. But always be a little suspicious. The techniques of power series were developed largely in the nineteenth century when no computers were available. For you, it's a different ball game! If you wonder if some calculations you have done are correct it may be very simple for you to write a little program to test things out. You may learn much more in this way than trying to rely on some nineteenth-century theorem.

Infinite Series Help Us With Nasty Functions And Local Calculus

We have much to learn about the mathematics of infinite series. Once you get into the spirit of it, it won't be too bad. Before we get into the more pedantic aspects of the subject, we are going to spend some time fooling around with power series approximations to functions. There are many possibilities here, but since we are fresh from studying integral calculus, we shall look at some applications of integration in the local calculus setting. In doing this, we shall work with nasty functions since nice functions can be treated by the methods of CHAPTER 4. One problem is to recognize the nasty as opposed to the nice functions. By "nasty" we mean "its integral is not expressible in terms of elementary functions." As you browse through the TABLE OF INTEGRALS in the appendix, you will see many horrible-looking functions that clever people have been able to integrate in terms of elementary functions. Still, you will notice that many reasonable-looking functions are missing from the table. Indefinite integrals involving $(a + bx)^{1/2}$ and $(a + bx^2)^{1/2}$ are given, but not of $(a + bx^3)^{1/2}$ or $(a + bx^4)^{1/2}$. Probably these latter two functions are nasty. Integrals of $\sin(\ln(x))$ and $\cos(\ln(x))$ are given but not of $\ln(|\sin(x)|)$ and $\ln(|\cos(x)|)$. Probably these are nasty. And where in the world are the integrals of $\sin(x^2)$ and $\cos(x^2)$? Also, no antiderivatives are given for e^{x^2} or e^{-x^2}. These are very important functions in statistics. Just as with humans, functions can be important even though they are nasty!

Following our usual pattern, we now give a series of exercises involving some nasty functions and local calculus. After the exercises, we give the solutions. Study them carefully. Then we give variations on these exercises for you to try.

Now We Try Some Local Calculus

5.1 EXERCISES

(1) Find the area under the graph of $f(x) = \dfrac{\sin(x)}{x}$ for $0 \leqslant x \leqslant \pi$.

(2) Find the volume of the solid of revolution obtained by revolving the curve $f(x) = \dfrac{\sin(x)}{x}$, $0 \leqslant x \leqslant \pi$, about the x-axis.

(3) Find the arclength of the curve $y(x) = x^3/3$ for x between $-1/2$ and $+1/2$.

5.2 SOLUTIONS TO EXERCISE 5.1

A Local Area Problem

(1) We are asked to compute $\displaystyle\int_0^{\pi} \frac{\sin(x)}{x}dx$. The first thing we do is write a little program in BASIC to get a feel for this function. Here is the program and its output. Study it carefully.

```
10  FOR X=.1 TO 3.1  STEP .1
20  PRINT X,SIN(X)/(X)
30  S=S+SIN(X)/X
40  NEXT X
50  PRINT "A = " S*.1
```

X	SIN(X)/X	X	SIN(X)/X
.1	.9983341	1.8	.5410264
.2	.9933466	1.9	.4980526
.3	.9850672	2	.4546486
.4	.9735459	2.1	.411052
.5	.9588511	2.2	.3674984
.6	.9410708	2.3	.3242197
.7	.920311	2.4	.281443
.8	.8966951	2.5	.239389
.9	.8703633	2.6	.1982699
1	.8414709	2.7	.158289
1.1	.8101885	2.8	.1196388
1.2	.7766993	2.9	.0825
1.3	.7411986	3.0	4.704026E-02
1.4	.7038926	3.1	1.341338E-02
1.5	.6649966	A = 1.802058	
1.6	.6247335		
1.7	.5833321	A=Approximate area=1.8 square units	

First Try A Riemann Sum, Then A Power Series

Notice that the values of $\dfrac{\sin(x)}{x}$ have been computed at intervals of 0.1. If we sum these values, all of which are positive, and multiply by the interval width of 0.1, we obtain a Riemann sum approximation to the integral. If you think about it a bit, you will see that this Riemann sum is going to be a little bit smaller than the actual area we are seeking. In any case, we get a rough idea that the area is about 1.8 square units. Knowing this may keep us from making fools of ourselves in the calculations to follow. Now let's try and compute the same area using our infinite series expansion for $\int \frac{\sin(x)}{x}dx$. We had found from the tables in the appendix that

$$\int \frac{\sin(x)}{x}dx = x - \frac{x^3}{3 \cdot 3!} + \frac{x^5}{5 \cdot 5!} - \frac{x^7}{7 \cdot 7!} + \frac{x^9}{9 \cdot 9!} \cdots$$

Let's take the polynomial $F(x) = p_7(x)$ as our approximation to the integral. Here is a BASIC program to compute $F(\pi) = F(3.14)$. We shall learn later that in a series with decreasing terms that alternate in sign, the maximum error is less than the absolute value of the first term omitted. We have used that fact to estimate the error in this BASIC program's approximation to the integral. The value of the area as estimated by F(3.14) is 1.84, which agrees pretty closely with our earlier Riemann sum approximation.

```
10  PRINT "ENTER X ":INPUT X
20  F3=3*2:F5=5*4*F3:F7=7*6*F5:F9=9*8*F7
30  P1=X:P3=X^3/(3*F3):P5=X^5/(5*F5):P7=X^7/(7*F7)
40  PRINT "F("X")=" P1-P3+P5-P7
50  PRINT "MAX ERROR IS ",X^9/(9*F9)

RUN
ENTER X
? 3.14
F( 3.14)=  1.843483
MAX ERROR IS    9.085761E-03
```

What If We Hadn't Found The Power Series In The Appendix?

In a certain sense, we lucked out in finding $\int \frac{\sin(x)}{x}dx$ expressed as an infinite series in our table. A more typical scenario would have been that we found only the series

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$$

We then would have divided all of the terms of this series by x to obtain

$$\frac{\sin(x)}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \ldots$$

Integrating this series term by term gives the series for $\int \frac{\sin(x)}{x} dx$. We shall learn later how to justify these steps. For now let's be daring! After all, we wrote some BASIC programs to give us some independent evidence of the validity of these calculations. Generally, if these types of calculations are unjustified, they produce real garbage that is easily detected by a little common sense.

A Local Volume Problem

(2) Using the methods of CHAPTER 4, we have that the volume is

$$V = \pi \int_0^{\pi} \frac{\sin^2(x)}{x^2} dx.$$

We begin with a short BASIC program that computes values of $\frac{\sin^2(x)}{x^2}$ at intervals of 0.1 and estimates the volume with a Riemann sum. The estimated volume is 4.3 cubic units. Here is the program and some output.

A Riemann Sum Approximation To The Local Volume Problem

```
10 FOR X=.1 TO 3.1 STEP .1
20 PRINT X, (SIN(X))^2/X^2
30 S=S+(SIN(X))^2/X^2
40 NEXT X
50 PRINT "INTEGRAL IS ABOUT "S*.1
60 PRINT "VOLUME IS ABOUT "3.14*S*.1
```

.1	.9966711	1.6	.3902919
.2	.9867376	1.7	.3402764
.3	.9703576	1.8	.2927095
.4	.9477916	1.9	.2480563
.5	.9193953	2	.2067054
.6	.8856142	2.1	.1689638
.7	.8469723	2.2	.135055
.8	.8040621	2.3	.1051184
.9	.7575323	2.4	7.921019E-02
1	.7080734	2.5	5.730708E-02

1.1	.6564054	2.6	3.931096E-02
1.2	.6032616	2.7	2.505542E-02
1.3	.5493753	2.8	1.431345E-02
1.4	.4954648	2.9	6.806249E-03
1.5	.4422204	3.0	2.212786E-03
		3.1	1.799188E-04
		INTEGRAL IS ABOUT	1.368151
		VOLUME IS ABOUT	4.295995

We Get A Chance To Compute A Power Series From Scratch

To apply power series methods to this problem, we need an infinite series for $\sin^2(x)$. No such series is given in the appendix. We could take the series for sin(x) and square it. A better approach for us at this point is to learn the general rule for forming power series expansions of a function about $x = 0$. The general method was described in CHAPTER 3, following the discussion of TAYLOR POLYNOMIALS. The general rule for any function h(x) is

$$h(x) = h(0) + h^{(1)}(0)x + \frac{h^{(2)}(0)}{2!}x^2 + \frac{h^{(3)}(0)}{3!}x^3 + \ldots + \frac{h^{(n)}(0)}{n!}x^n + \ldots$$

In this expression, $h^{(n)}(x)$ is the n^{th} derivative of h(x). This n^{th} derivative evaluated at $x = 0$ is $h^{(n)}(0)$. Now let's compute some derivatives of $h(x) = \sin^2(x)$:

$$h^{(1)}(x) = 2\sin(x)\cos(x) = \sin(2x)$$

$$h^{(2)}(x) = 2\cos(2x)$$

$$h^{(3)}(x) = -2^2\sin(2x)$$

$$h^{(4)}(x) = -2^3\cos(2x)$$

$$h^{(5)}(x) = +2^4\sin(2x)$$

$$h^{(6)}(x) = +2^5\cos(2x)$$

$$h^{(7)}(x) = -2^6\sin(2x)$$

$$h^{(8)}(x) = -2^7\cos(2x)$$

$$h^{(9)}(x) = +2^8\sin(2x)$$

Do you see the pattern? Note that $f^{(k)}(0) = 0$ for $k = 0, 1, 3, 5, 7, \ldots$. Thus we obtain

$$\sin^2(x) = +\frac{2^1}{2!}x^2 - \frac{2^3}{4!}x^4 + \frac{2^5}{6!}x^6 - \frac{2^7}{8!}x^8 + \frac{2^9}{10!}x^{10} - \ldots$$

Now we divide this series for $\sin^2(x)$ by x^2 to obtain

$$\frac{\sin^2(x)}{x^2} = +1 - \frac{2^3}{4!}x^2 + \frac{2^5}{6!}x^4 - \frac{2^7}{8!}x^6 + \frac{2^9}{10!}x^8 - \dots$$

Integrating term by term, we obtain

$$\int \frac{\sin^2(x)}{x^2}dx = +x - \frac{2^3}{3 \cdot 4!}x^3 + \frac{2^5}{5 \cdot 6!}x^5 - \frac{2^7}{7 \cdot 8!}x^7 + \frac{2^9}{9 \cdot 10!}x^9 - \dots$$

You should notice the powers of 2 that appear in numerators of the coefficients in this series. They are going to make convergence slow for any large values of x. Even for x = 3.14 it is necessary to take the following polynomial to get reasonable accuracy:

$$p_{13}(x) = x - \frac{2^3}{3 \cdot 4!}x^3 + \frac{2^5}{5 \cdot 6!}x^5 - \frac{2^7}{7 \cdot 8!}x^7 + \frac{2^9}{9 \cdot 10!}x^9 - \frac{2^{11}}{11 \cdot 12!}x^{11} - \frac{2^{13}}{9 \cdot 14!}x^{13}.$$

The following BASIC program confirms this and computes directly $p_{13}(3.14)$. The volume estimated by this program is 4.46, which is pretty close to our Riemann sum method above. This program is a simple-minded and direct translation of the formula for $p_{13}(x)$ into BASIC. It would be much better numerically to compute the factorials F4, F6, etc., and the numbers P1, P3, etc., recursively within a loop. This is another story, however. Remember, we are just fooling around with series at this point.

A BASIC Program To Solve The Local Volume Problem

```
LIST
10  PRINT "ENTER X ":INPUT X
20  F4=24:F6=6*5*F4:F8=8*7*F6:F10=10*9*F8:F12=
    12*11*F10:F14=14*13*F12
30  P1=X:P3=(2*X)^3/(3*F4):P5=(2*X)^5/(5*F6):P7=
    (2*X)^7/(7*F8):P9=(2*X)^9/(9*F10):P11=(2*X)^(11/(11*F12):P13=
    (2*X)^13/(13*F14)
40  INTEGRAL = P1-P3+P5-P7+P9-P11+P13
50  PRINT "INTEGRAL IS ABOUT "INTEGRAL
60  PRINT "V("X") IS ABOUT "3.14*INTEGRAL
70  F16=16*14*F14
80  PRINT "MAXIMUM ERROR IS "(2*X)^15/(15*F16)
```

```
RUN
ENTER X
? 3.14
INTEGRAL IS ABOUT    1.420813
V( 3.14 ) IS ABOUT   4.461352
MAXIMUM ERROR IS   3.181611E-03
```

Finally, A Local Arclength Problem!

(3) Using the techniques of CHAPTER 4, we have

$$\text{ARCLENGTH} = \int_{-1/2}^{1/2} (1 + (y'(x))^2)^{1/2}dx = \int_{-1/2}^{1/2} (1 + x^4)^{1/2}dx$$

$$= 2\int_{0}^{1/2} (1 + x^4)^{1/2}dx.$$

We look in the appendix and find that

$$(1 + x)^{1/2} = 1 + \frac{1}{2}x - \frac{1 \cdot 1}{2 \cdot 4}x^2 + \frac{1 \cdot 1 \cdot 3}{2 \cdot 4 \cdot 6}x^3 - \frac{1 \cdot 1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 8}x^4 + \ldots$$

Substituting x^4 for x we obtain

$$(1 + x^4)^{1/2} = 1 + \frac{1}{2}x^4 - \frac{1 \cdot 1}{2 \cdot 4}x^8 + \frac{1 \cdot 1 \cdot 3}{2 \cdot 4 \cdot 6}x^{12}$$

$$- \frac{1 \cdot 1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 8}x^{16} + \ldots$$

Integrating term by term, we find

$$\int (1 + x^4)^{1/2}dx = x + \frac{1}{2}\frac{x^5}{5} - \frac{1 \cdot 1}{2 \cdot 4}\frac{x^9}{9} + \frac{1 \cdot 1 \cdot 3}{2 \cdot 4 \cdot 6}\frac{x^{13}}{13} - \ldots$$

If we set x = 1/2 in this expression we should get a good approximation to half the arclength. Actually, we need only the first three terms of this series. With x = 1/2, we again have a series with decreasing terms of alternating sign. Thus the error is no worse than the first term omitted. The following little BASIC program shows us the error for omitting the term involving x^{13}:

A Basic Program For The Error

```
LIST
10  PRINT "INPUT X ":INPUT X
20  PRINT (3/(2*4*6*13))*X^(13)

RUN
INPUT X
? .5
 5.868765E-07
```

The next program computes the first three terms of the series evaluated at $x = 1/2$.

A Basic Program For The Arclength

```
LIST
10  PRINT "ENTER X ":INPUT X
20  P1=X:P5=X^5/2*5:P9=X^9/2*4*9
30  PRINT "F("X") = "P1+P5-P9
40  PRINT "ARCLENGTH IS ABOUT "2*(P1+P5-P9)

RUN
ENTER X
? .5
F( .5) =  .5429688
ARCLENGTH IS ABOUT  1.085938
```

Something We've Swept Under The Rug: Intervals Of Convergence

The arclength is thus about 1.086. There is an important detail that we have swept under the rug thus far in our calculations with the series for $(1 + x)^{1/2}$. We found the series expansion for $(1 + x)^{1/2}$ in the math tables in the appendix under SERIES AND PRODUCTS. Just after the series expansion for $(1 + x)^{1/2}$ you see the notation $[x^2 < 1]$. This notation means that the series is valid only for values of x such that $x^2 < 1$. In other words, $|x| < 1$ or, equivalently, $-1 < x < +1$. The $\{x: -1 < x < +1\}$ is also denoted by $(-1, +1)$ and is called the "interval of convergence of the series." We will learn more about intervals of convergence (THEOREM 5.57). The main thing we need to note is that $x = 1/2$ is in the interval of convergence and so is $x^4 = (1/2)^4 = 1/16$. If you look at the series for sin(x) in the appendix, you will see that the interval of convergence is $[x^2 < \infty]$.

The series for sin(x) converges for all x. This means that no matter how large x is, if you take enough terms of the series for sin(x) you will eventually get a good approximation. The number of terms required may be very large, however! In the case of the series for $(1 + x)^{1/2}$, no matter how many terms you take, you won't get good approximations to values of x outside the interval of convergence (values such as $x = 2$ or $x = 3$, etc.). The general rule for integrating a series term by term to obtain a series that approximates the integral is that the limits of integration must both lie in the interval of convergence. Thus, in our case, the limits were 0 and 1/2, which both lie in the interval of convergence of $(1 + x)^{1/2}$ and hence of the series for $(1 + x^4)^{1/2}$, which also has interval of convergence $-1 < x < +1$. Does all this talk of intervals of convergence make you nervous about the validity of our calculation of the arclength of $x^3/3$ from $-1/2$ to $1/2$? If so that's a healthy sign. Let's check out our calculations with a little BASIC program:

Riemann Sum Approximation For Arclength

```
LIST
10 FOR X=0 TO .5 STEP .01
20 S=S+(1+X^4)^.5
30 NEXT X
40 PRINT "THE INTEGRAL IS ABOUT "2*S*.01

RUN
THE INTEGRAL IS ABOUT  1.026509
```

The Riemann sum approximation is close to the result obtained from the infinite series, so probably things are all right.

There is one thing left that we should do in connection with this arclength problem. We should derive the series for $(1 + x)^{1/2}$. Remember the general rule

$$y(x) = y(0) + \frac{y^{(1)}(0)}{1!}x + \frac{y^{(2)}(0)}{2!}x^2 + \frac{y^{(3)}(0)}{3!}x^3 \ldots$$

Computing some derivatives, we find $y^{(1)}(x) = (1/2)(1 + x)^{-1/2}$, $y^{(2)}(x) = (1/2)(-1/2)(1 + x)^{-3/2}$, $y^{(3)}(x) = (1/2)(-1/2)(-3/2)(1 + x)^{-5/2}$, $y^{(4)}(x) = (1/2)(-1/2)(-3/2)(-5/2)(1 + x)^{-7/2}$, ... Substituting $x = 0$ into these expressions and putting the resulting numbers into the expansion for y(x) gives the series in the appendix for $(1 + x)^{1/2}$. You should carry out this calculation carefully. Only the interval of convergence remains a mystery!

Now You Get To Try Some Local Calculus!

We now give a series of variations on EXERCISE 5.1. The problems in these variations correspond closely to their counterparts in EXERCISE 5.1, so you may want to check SOLUTIONS 5.2 periodically.

5.3 VARIATIONS ON EXERCISE 5.1

(1) Find the area under the graph of $f(x) = \dfrac{\sin(x/2)}{x}$, $0 \leqslant x \leqslant \pi$. Sketch the graph of f(x).

(2) Find the volume of the solid of revolution obtained by revolving the curve $f(x) = \dfrac{\sin(x/2)}{x}$, $0 < x < \pi$, about the x-axis.

(3) Find the arclength of the curve $y(x) = x^3/3$ for x between -0.9 and $+0.9$. Make sure the error is less than .001. This problem would be much harder if we asked for the arclength from -2 to $+2$. Do you see why?

Power Series Are Better Than Riemann Sums When Parameters Occur

5.4 VARIATIONS ON EXERCISE 5.1

(1) Find the area under the graph of $f_\beta(x) = \dfrac{\sin(\beta x)}{x}$, $0 \leqslant x \leqslant \pi$, as a function of β, $0 \leqslant \beta \leqslant 1$. Sketch the graph of $f_\beta(x)$ for several values of β.

(2) Find, as a function of β, the volume of the solid of revolution obtained by revolving the curve $f_\beta(x) = \dfrac{\sin(\beta x)}{x}$, $0 \leqslant x \leqslant \pi$, about the x-axis. Again, assume $0 \leqslant \beta \leqslant 1$.

(3) Find the arclength of the curve $y(x) = (\alpha x)^3/3$, $0 \leqslant \alpha \leqslant 3/2$, for x between -0.5 and $+0.5$. Your answer should be a function of α such that the error is less than .001 for all values of α with $0 \leqslant \alpha \leqslant 3/2$.

A Different Nasty Function—Same Techniques

5.5 VARIATIONS ON EXERCISE 5.1

(1) Find the area under the graph of $f(x) = \sin(x^2)$, $0 \leqslant x \leqslant (\pi)^{1/2}$. Sketch the graph of f(x).

(2) Find the volume of the solid of revolution obtained by revolving the curve $f(x) = \sin(x^2)$, $0 \leq x \leq (\pi)^{1/2}$, about the x-axis.

(3) Find the arclength of the curve $y(x) = x^3$ for x between -0.5 and $+0.5$. Make sure the error is less than .001.

By now you should have some idea how infinite series can be used to extend the range of applicability of calculus. It's now time to be a little more systematic about our study of infinite series.

We Start With Sequences

5.6 DEFINITION: SEQUENCES Let $Z = \{0, 1, 2, \ldots\}$ denote the set of nonnegative integers. A function f whose domain D is a subset of Z and whose range is the set R of real numbers is called a real valued *sequence*. If D has infinitely many elements, then f is called an infinite sequence.

Playing The Envelope Game With Sequences

It's time for the ENVELOPE GAME again. Imagine you have an envelope. Inside is a sequence. What are you going to see when you open the envelope? Well, you might see some ordered pairs written on a piece of old yellow parchment: (0, 5.45), (1, 6.43), (5, 3.45). This would be the sequence with $D = \{0, 1, 5\}$. At 0, this sequence would have the value 5.45, at 1 the value 6.43, and 5 the value at 3.45. If the sequence is an infinite sequence, you won't see all of its values written down (obviously!). You might see something like (0, 0), (1, 2) (2, 4), (3, 6), (4, 8), ... The ". . . ," read "dot, dot, dot," is meant to imply that anyone should be able to figure out the general rule from what is given: (n, 2n). Another way to describe the same sequence is $D = Z$, $f(n) = 2n$. Another way is $D = Z$, $f(n) = a_n$ where $a_n = 2n$. Another way is $D = Z$, $a_n = 2n$. Another way is $a_n = 2n$, $n = 0, 1, 2, \ldots$. In all cases, we must be clear about D and about the rule which assigns to each element of D a real number.

The next definition will sound strange at first, but you will get used to it, if not learn to love it.

Epsilons And Limits

5.7 DEFINITION: LIMIT OF A SEQUENCE Let a_n, $n = 0, 1, 2, 3, \ldots$ be an infinite sequence. We say that a real number A is the limit of a_n as n goes to infinity and write

$$\operatorname*{limit}_{n\to\infty} a_n = A$$

if for every real number $\epsilon > 0$ there exists an integer N_ϵ such that for all $n > N_\epsilon$, $|a_n - A| < \epsilon$.

You Give Me ϵ Then I Give You N_ϵ Such That . . .

Many students find DEFINITION 5.7 annoying. Consider, for example, the sequence $a_n = (2n + 1)/(n + 1)$. As n goes to infinity this sequence obviously approaches $A = 2$ as a limit. We don't need DEFINITION 5.7 to see that this is true, so why confuse the obvious? The reason that we need DEFINITION 5.7 is that it provides a necessary technical tool for discussing limits of sequences in general terms, apart from any specific examples. If DEFINITION 5.7 seems confusing to you, think of it as sort of a game. Imagine that you are in a room sitting at a desk. On the desk is a piece of paper with a sequence a_n, $n = 0, 1, 2, \ldots$, that you've announced converges to a number A. Every now and then, at random intervals, someone opens the door to the room and hands you a positive real number ϵ (like $\epsilon = .001$, for example). You have to give that person an integer N_ϵ such that for all $n > N_\epsilon$, $|a_n - A| < \epsilon$. If you can prove that you can do this for any ϵ that may be given to you, then that proves that A is the limit of the sequence a_n. For the sequence $a_n = (2n + 1)/(n + 1)$, if you are given $\epsilon > 0$, you can take N_ϵ to be any integer greater than or equal to ϵ^{-1}. If $n > N_\epsilon \geq \epsilon^{-1}$ then

$$|a_n - A| = \left|\frac{2n + 1}{n + 1} - 2\right| = \left|\frac{-1}{n + 1}\right| < \frac{1}{n} < \frac{1}{N_\epsilon} \leq \epsilon.$$

This proves, using DEFINITION 5.7, the obvious fact that $(2n + 1)/(n + 1)$ converges to $A = 2$. If you take $\epsilon = .001$ in the above inequality, then $\epsilon^{-1} = 1000$ and $N_{.001}$ can be any integer greater than or equal to 1000. In fact, $N_{.001} = 1000$ works fine. Thus, for all $n > 1000$,

$$|a_n - A| = \left|\frac{2n + 1}{n + 1} - 2\right| < .001.$$

It is more important that the beginning calculus student develop a strong intuitive feeling for limits than a technical ability to work with DEFINITION 5.7. A little awareness of the latter is all that we ask at this point. Actually, if you think about it a bit, DEFINITION 5.7 has a strong intuitive appeal. It says that if you claim that the sequence a_n approaches A, then, given *any level* of accuracy ϵ, which we think of as a small number, you must be able to specify an integer N_ϵ such that past N_ϵ the sequence gets *and stays* within that level of accuracy from A.

If It Doesn't Converge, It Diverges

If a sequence a_n has a limit in the sense of DEFINITION 5.7, it is called a *convergent sequence*. A sequence a_n, $n = 0, 1, 2, \ldots$, that does not have a limit is called divergent. A simple example of a divergent sequence is the sequence $a_n = (-1)^n$. This sequence hops back and forth from $+1$ to -1. Given any $\epsilon < 2$, it is obviously impossible to find the N_ϵ demanded by DEFINITION 5.7. Another divergent sequence is the sequence $a_n = n$, $n = 0, 1, 2, \ldots$. The A of DEFINITION 5.7 is specified to be a real number. This sequence $a_n = n$ will never get close and stay close to any real number A, because a_n just gets larger and larger as n gets larger and larger. We say that the sequence a_n is "unbounded."

5.8 DEFINITION A sequence a_n, $n = 0, 1, 2, \ldots$, is *bounded* if there is a positive number B such that $|a_n| < B$ for all n.

5.9 THEOREM A convergent sequence is bounded.

Proof: Let a_n, $n = 0, 1, 2, \ldots$, be a sequence with

$$\lim_{n\to\infty} a_n = A.$$

Taking $\epsilon = 1$ in DEFINITION 5.7, let N_1 be such that for $n > N_1$, $|a_n - A| < 1$. Let B be the maximum of $|a_0|, |a_1|, |a_2|, \ldots, |a_{N_1}|, |A| + 1$. Then $|a_n| \leq B$ for all n. This completes the proof.

It's Equivalent To Its Contrapositive

THEOREM 5.9 says that if "a_n, $n = 0, 1, 2, \ldots$, is convergent" then "a_n, $n = 0, 1, 2, \ldots$, is bounded." This is equivalent to saying if "a_n, $n = 0, 1, 2, \ldots$, is not bounded" then "a_n, $n = 0, 1, 2, \ldots$, is divergent." These two statements are called *contrapositive* statements. If P and Q are propositions, then the statement "if P then Q" is equivalent to "if not Q then not P." In our example, P = "a_n, $n = 0, 1, 2, \ldots$, is convergent" and Q = "a_n, $n = 0, 1, 2, \ldots$, is bounded." Here is an example from real life: if "there is a cow in the barn" then "there is a mammal in the barn." The contrapositive statement, which is logically equivalent, is if "there is no mammal in the barn'' then ''there is no cow in the barn.'' Care must be taken with ''real life'' interpretations of the contrapositive.

The Converse Is Not The Same As The Contrapositive

You should pay careful attention to the distinction made in mathematical proofs between the ''contrapositive'' and the ''converse.'' The converse of the statement ''if P then Q'' is the statement ''if Q then P.'' These statements are definitely not logically equivalent. Each must be proved or disproved separately. In the statement if ''a_n, n = 0, 1, 2, . . ., is convergent'' then ''a_n, n = 0, 1, 2, . . ., is bounded,'' the converse is if ''a_n, n = 0, 1, 2, . . ., is bounded'' then ''a_n, n = 0, 1, 2, . . ., is convergent.'' This latter statement is false, as the example $a_n = (-1)^n$, n = 0, 1, 2, . . ., shows. If both the statement ''if P then Q'' and its converse ''if Q then P'' are true, then we say ''P if and only if Q'' is true. As an example, suppose we are talking about triangles in a trigonometry course and the lengths of the sides of a triangle T are denoted by $a \leq b \leq c$. Let P = ''T is a right triangle'' and Q = ''$a^2 + b^2 = c^2$.'' The statement ''if P then Q'' is the PYTHAGOREAN THEOREM. The statement ''if Q then P'' is also a true theorem, a consequence of the law of cosines. Thus, the statement ''P if and only if Q'' is valid.

We now state some of the basic rules for operating with limits of sequences. These rules are a special case of the RULES FOR LIMITS (3.14).

Some More Rules And The Agony Of A Proof

5.10 THEOREM: RULES FOR LIMITS OF SEQUENCES Suppose that a_n, n = 0, 1, 2, . . ., and b_n, n = 0, 1, 2, . . ., are convergent infinite sequences. Let

$$\operatorname*{limit}_{n\to\infty} a_n = A \text{ and } \operatorname*{limit}_{n\to\infty} b_n = B.$$

Define sequences t_n, s_n, p_n, and q_n, n = 0, 1, 2, . . ., by $t_n = \alpha a_n$, where α is a real number, $s_n = a_n + b_n$, and $p_n = a_n b_n$. If $b_n \neq 0$ for all n, define $q_n = a_n/b_n$. Then

(1) $\operatorname*{limit}_{n\to\infty} t_n = \alpha A$

(2) $\operatorname*{limit}_{n\to\infty} s_n = A + B$

(3) $\lim_{n \to \infty} p_n = AB$

(4) $\lim_{n \to \infty} q_n = A/B$ if $B \neq 0$.

Proof: We give the proofs of (2) and (3). You should try to do similar proofs for (1) and (4). We use DEFINITION 5.7. To prove (2) we must show that given any $\epsilon > 0$ we can find an integer N_ϵ such that, for all $n > N_\epsilon$, $|s_n - (A + B)| = |a_n + b_n - A - B| < \epsilon$. We shall use the fact that for any real numbers x and y, $|x + y| \leq |x| + |y|$. Given ϵ, let $\rho = \epsilon/2$. From DEFINITION 5.7, we know we can find N_ρ such that, for all $n > N_\rho$, $|a_n - A| < \rho$. Likewise, we can find N_ρ (perhaps different than the one of the last sentence) such that, for all $n > N_\rho$, $|b_n - B| < \rho$. Take N_ϵ to be the larger of these two N_ρ. Thus, for $n > N_\epsilon$, we have

$$|s_n - (A + B)| = |a_n + b_n - A - B|$$

$$\leq |a_n - A| + |b_n - B| < \rho + \rho = \epsilon.$$

This completes the proof of (2). To prove (3), we must show that given any $\epsilon > 0$, we can find an integer N_ϵ such that, for all $n > N_\epsilon$, $|a_n b_n - AB| < \epsilon$. We have

$$|a_n b_n - AB| = |(a_n - A)b_n + (b_n - B)A|$$

$$\leq |a_n - A|\,|b_n| + |b_n - B|\,|A|.$$

By THEOREM 5.9, the sequence b_n is bounded by some number M, which we choose to be larger than $|A|$. Now, let $\rho = \epsilon/(2M)$. Then, by DEFINITION 5.7, there is an integer N_ρ such that, for all $n > N_\rho$, $|a_n - A| < \rho$. Likewise, there is an integer N_ρ (perhaps different from the one of the last sentence) such that, for all $n > N_\rho$, $|b_n - B| < \rho$. Take N_ϵ to be the larger of these two N_ρ. Then, for all $n > N_\epsilon$,

$$|a_n b_n - AB| \leq |a_n - A|\,|b_n| + |b_n - B|\,|A|$$

$$\leq |a_n - A|\,M + |b_n - B|\,M < \rho M + \rho M = \epsilon.$$

This completes the proof of (3) of THEOREM 5.10.

Look again at THEOREM 5.9 and DEFINITION 5.8. A sequence a_n, $n = 0, 1, 2, \ldots$, is "bounded" by B if $|a_n| < B$ for all n. We can now refine that idea a bit.

Upper And Lower Bounds And A Fundamental Axiom

5.11 DEFINITION Let a_n, $n = 0, 1, 2, \ldots$, be a sequence. We say that the real number U is an *upper bound* for the sequence if $a_n < U$ for all n. We say the real number L is a *lower bound* for the sequence if $a_n > L$ for all n. A real number LU is the *least upper bound* of the sequence a_n if it is an upper bound for a_n and if, for any real number $x < LU$, x is not an upper bound of a_n. A real number GL is the *greatest lower bound* of the sequence a_n if it is a lower bound for a_n and if, for any real number $x > GL$, x is not a lower bound of a_n.

Let's look at a couple of examples of DEFINITION 5.11. As a first example, take $f(x) = -(x - 3)^2 + 4$. Define $a_n = f(n)$, $n = 0, 1, 2, \ldots$. Then, since the maximum value of f(x) is +4 at x = +3, any number $U > 4$ is an upper bound for this sequence. The number 4 is the least upper bound because any number $x < 4$ has $a_3 > x$ and hence x is not an upper bound for the sequence a_n. As a second example, take $g(x) = 4x^2/(x^2 + 5)$ and define $b_n = g(n)$, $n = 0, 1, 2, \ldots$. By computing $g'(x) = 40x/(x^2 + 5)^2$ we see that g(x) is increasing and hence the sequence b_n satisfies $b_n < b_{n+1}$ for all $n = 0, 1, 2, \ldots$. It is easy to see the limit as n approaches infinity of b_n is 4. Thus 4 is an upper bound. In fact, 4 is the least upper bound. See if you can convince yourself why this is true! It is a special case of THEOREM 5.13 below.

It is a basic axiom of the real number system that every sequence of real numbers that has an upper bound U must have a unique least upper bound LU. Likewise, every sequence of real numbers that has a lower bound must have a unique greatest lower bound GL. For certain types of sequences, these unique bounds are also the limits.

Bounded Monotone Sequences Always Converge

5.12 DEFINITION A sequence a_n, $n = 0, 1, 2, \ldots$, is nondecreasing if $a_n \leqslant a_{n+1}$ for $n = 0, 1, 2, \ldots$. A sequence a_n, $n = 0, 1, 2, \ldots$, is nonincreasing if $a_n \geqslant a_{n+1}$ for $n = 0, 1, 2, \ldots$.

5.13 THEOREM Every bounded nondecreasing sequence converges to its least upper bound LU. Every bounded nonincreasing sequence converges to its greatest lower bound GL.

Proof: Suppose that a_n, $n = 0, 1, 2, \ldots$, is nondecreasing and bounded. In particular, a_n must be bounded above and hence must have a least upper bound LU. Given $\epsilon > 0$, we note that the real number $x = LU - \epsilon$ is not an upper bound for the sequence a_n by the definition of the least upper bound LU. Thus, there is an integer m such that $a_m > LU - \epsilon$. Of course, $a_m \leq LU$ as LU is an upper bound for the whole sequence a_n. If we take $N_\epsilon = m$, then, by the fact that the sequence a_n is nondecreasing, for all $n > N_\epsilon$, $LU - \epsilon < a_m \leq a_n \leq LU$. In other words, for $n > N_\epsilon$, $|a_n - LU| < \epsilon$. Thus, by DEFINITION 5.7, LU is the limit of the sequence a_n. This completes the proof of THEOREM 5.13 (the proof of the second statement of the threorem is directly analogous).

Diverging To Plus Or Minus Infinity

In DEFINITION 5.7, we defined what it meant for a sequence to converge to a limit A, where A is a real number. We stated that a sequence that did not converge was called a *divergent sequence*. Divergent sequences can be bounded, such as the sequence $(-1)^n$, $n = 0, 1, 2, \ldots$, or unbounded, such as $a_n = n$, $n = 0, 1, 2, \ldots$, or $a_n = -n$, $n = 0, 1, 2, \ldots$, or $a_n = (-1)^n n$, $n = 0, 1, 2, \ldots$. In the sequence $a_n = n$, the terms get steadily larger and larger without any upper bound. In the sequence $a_n = -n$, the terms get steadily smaller and smaller without any lower bound. In the first case, we say that the sequence "diverges to plus infinity" or "tends to plus infinity." In the second case, we say the sequence "diverges to minus infinity" or "tends to minus infinity." Although it is technically an abuse of the notation to do so, we sometimes write

$$\operatorname*{limit}_{n \to \infty} a_n = +\infty \qquad \text{and} \qquad \operatorname*{limit}_{n \to \infty} a_n = -\infty$$

to describe these two situations. The sequence $(-1)^n n$ doesn't tend to either plus infinity or minus infinity. It hops back and forth between larger and larger positive and negative values. We simply call this sequence an "unbounded divergent sequence." You should be getting sophisticated enough by now to understand the following definition:

5.14 DEFINITION Let a_n, $n = 0, 1, 2, \ldots$, be a sequence. We say that a_n *diverges to plus infinity* if for all $x > 0$, there exists an integer N_x such that for all $n > N_x$, $a_n > x$. Similarly, we say that a_n diverges to minus infinity if for all $x < 0$, there exists an integer N_x such that for all $n > N_x$, $a_n < x$.

The intuitive idea of DEFINITION 5.14 is that if the sequence a_n diverges to plus infinity, then given any number x, which we imagine to be very large, we can always find some point in the sequence beyond which *all* of the terms are larger than x.

Some Practice With Sequences

In the next exercise, you are asked to discuss the divergence or convergence of certain sequences. This means that you should state whether or not the sequence diverges or converges. If it converges, try to find the limit. If it diverges, state whether or not the sequence is bounded. If the sequence is not bounded, state whether or not the sequence diverges to plus infinity or to minus infinity or neither.

5.15 EXERCISES

(1) Discuss the convergence or divergence of the following sequences:

(a) $\dfrac{2n^3 + 3n + 1}{3n^3 + 2}$, $n = 0, 1, 2, \ldots$

(b) $\dfrac{-n^3 + 1}{2n^2 + 3}$, $n = 0, 1, 2, \ldots$

(c) $\dfrac{(-n)^{n+1} + 1}{n^n + 1}$, $n = 0, 1, 2, \ldots$

(d) $\dfrac{n^n}{e^n}$, $n = 1, 2, \ldots$

(e) $\dfrac{(-n)^3 + 1}{n^3 + \ln(n)}$, $n = 0, 1, 2, \ldots$

(f) $\cos(\pi n)$, $n = 0, 1, 2, \ldots$

(g) $\dfrac{\log_2(n)}{\log_3(n)}$, $n = 2, 3, \ldots$

(h) $\dfrac{\log_2(n)}{n^{0.1}}$, n = 1, 2, . . .

(i) $\dfrac{\log_2(\log_2(n))}{\log_2(n)}$, n = 2, 3, . . .

(j) cos(n), n = 0, 1, 2, . . .

(k) $\left(1 + \dfrac{1}{n}\right)^n$, n = 1, 2, . . .

(2) By using a computer to generate these sequences, discuss their convergence or divergence:

(a) $\cos(n^2)$, n = 0, 1, 2, . . .

(b) s_n, n = 1, 2, . . ., where $s_{n+1} = (1/n)^{2^{-n}} s_n$ and $s_1 = 1$

(c) s_n, n = 1, 2, . . ., where $s_{n+1} = \left(1 + \dfrac{1}{n}\right)^{2^{-n}} s_n$ for n = 2, 3, . . ., and $s_1 = 1$

Study The Solution, Then Modify The Original

We now discuss the solutions to EXERCISE 5.15. Remember, after learning the solution, go back to the original problem, change it a little bit, and rework it. After doing this for all problems in EXERCISE 5.15, you should move on to the VARIATIONS ON EXERCISE 5.15.

5.16 SOLUTIONS TO EXERCISE 5.15

(1)(a) This sequence is a ratio of two polynomials in n (i.e., a rational function of n). If we divide numerator and denominator by the highest power of n that appears in the denominator, we can write the same sequence, except for the first term, as

$$\frac{2 + (3/n^2) + (1/n^3)}{3 + (2/n^3)}, \quad n = 1, 2, \ldots$$

In this form, it is obvious that the sequence converges to the limit 2/3 as n goes to infinity.

(1)(b) Again, we have a rational function of n. Divide by the highest power of n that appears in the denominator to obtain

$$\frac{-n + (1/n^2)}{2 + (3/n)}, \ n = 1, 2, \ldots$$

In this form, it is evident that the sequence diverges to minus infinity.

(1)(c) Again, we divide by the highest power of n in the denominator to obtain

$$\frac{-n(-1)^n + n^{-n}}{1 + n^{-n}}.$$

This is an unbounded, divergent sequence that oscillates between very large negative and positive values.

(1)(d) This is the same as the sequence $(n/e)^n$, which obviously diverges to plus infinity.

(1)(e) Dividing numerator and denominator by n^3 gives

$$\frac{-1 + 1/n^3}{1 + (\ln(n))/n^3}.$$

If you happen to know that the limit as n tends to infinity of $(\ln(n)/n^3$ is zero, then you will see that the limit of this expression is obviously -1. To understand this fact, we use L'HOPITAL'S RULE 3.15 to write

$$\lim_{n\to\infty} \frac{\ln(n)}{n^3} = \lim_{x\to\infty} \frac{\ln(x)}{x^3} = \lim_{x\to\infty} \frac{1}{3x^3} = 0.$$

(1)(f) This is the sequence $+1, -1, +1, \ldots$ which is bounded, divergent.

(1)(g) If we write $\log_3(n) = \log_3(2)\log_2(n)$, we see that every term in the sequence is $(\log_3(2))^{-1}$ and thus the sequence converges to this constant.

(1)(h) Using L'HOPITAL'S RULE again, we obtain

$$\lim_{n\to\infty} \frac{\log_2(n)}{n^{0.1}} = \lim_{x\to\infty} \frac{\log_2(x)}{x^{0.1}} = \lim_{x\to\infty} \frac{1}{(0.1)\ln(2)x^{0.1}} = 0.$$

Thus the limit of this sequence is zero. If you worked part (b) of EXERCISE 3.20(3) you will recall the useful fact that the limit as x goes to infinity of $(\ln(x))/x^a$ is zero for any $a > 0$.

(1)(i) This limit is zero. The reason is again the fact that the limit as x goes to infinity of $(\ln(x))/x^a$ is zero for any $a > 0$. In this case, we take $a = 1$ and use the fact that as x goes to infinity so does $\log_2(x)$.

(1)(j) This problem is a little trickier than it seems. If you write the following little BASIC program

```
10  PRINT COS(N)
20  N=N+1
30  GOTO 10
```

and watch the numbers stream out, it is clear that cos(n) does not converge. This is the correct intuition. The function cos(x) is +1 at all $x = 2\pi n$, n an integer. At all real numbers of the form $x = 2\pi n + \pi$, cos(x) is -1. Think about the intervals $[2\pi n - .5, 2\pi n + .5]$. In every such interval there must be an integer m. This means that $\cos(m) > \cos(.5)$. In every interval of the form $[2\pi n + \pi - .5, 2\pi n + \pi + .5]$ there must be an integer m′ for which $\cos(m') < \cos(\pi - 0.5) = -\cos(0.5)$. Thus, for infinitely many integers n, $\cos(n) > \cos(.5)$ and for infinitely many integers n, $\cos(n) < -\cos(0.5)$. This proves what common sense tells us, the sequence cos(n), n = 0, 1, 2, . . ., does not converge. It is bounded and divergent.

(1)(k) Let's write a little BASIC program to check this sequence out.

```
10  PRINT (1+ 1/N)^N
20  N=N+1
30  GOTO 10
         .
         .
         .
    2.717815
    2.717536
    2.717995
    2.717692
    2.718148
```

Amazing! It looks like this sequence is converging to the number e = 2.718 . . .. The easiest way to see this is to investigate the limit as n goes to infinity of $\ln\left(1 + \frac{1}{n}\right)^n = n\ln\left(1 + \frac{1}{n}\right)$. Using good old L'HOPITAL'S RULE again gives

$$\lim_{x\to\infty} x\ln(1 + x^{-1}) = \lim_{x\to\infty} \frac{\ln(1 + x^{-1})}{x^{-1}} = \lim_{x\to\infty} \frac{1}{1 + x^{-1}} = 1.$$

Thus, for the sequence $a_n = \left(1 + \frac{1}{n}\right)^n$, we have shown that the limit($\ln(a_n)$) = 1. This implies that limit(a_n) = e^1 = e. In general, if limit($\ln(a_n)$) = A > 0 then limit(a_n) = e^A. Formally, this is because the function ln(x) is one-to-one (FIGURE 2.13) and continuous (DEFINITION 3.12). It is also intuitively obvious if you take a look at the graph of ln(x) shown in FIGURE 2.13.

Now The Solutions To Exercise 5.15(2)

(2)(a) The method used to analyze problem 1(j) above doesn't work here. The cos(x) function is evaluated at the integer points 1, 4, 16, . . ., n^2, ... If $\cos(n^2)$ were to have a limit A as n goes to infinity, then the points of the sequence n^2, n = 1, 2, . . ., would have to eventually cluster about points in the set {x:cos(x) = A}. This seems highly unlikely, and can, with a little more effort than we want to make at this point, be proved not to happen. Here is a BASIC program with some sample output from the screen after the program has run for several minutes.

```
10  PRINT COS(N*N)
20  N=N+1
30  GOTO 10
            .
            .
            .
     .1177025
    -.9925256
     .9545957
     .9154154
     .6455998
    -.5313178
    -.9999428
    -.5312158
     .6454156
     .9155609
     .9544519
    -.992599
     .1169841
    -.3111826
    -.3815438
    -.7837191
    -.8679878
```

It is apparent that this sequence is bounded but divergent.

(2)(b) A sequence s_n defined in this manner is said to be "defined recursively." Note that the ratio s_{n+1}/s_n is less than 1. Thus the sequence s_n is nonincreasing and, since 0 is a lower bound for the sequence, it must converge to its greatest lower bound. The following program, when allowed to run a bit, produces the output shown.

```
10  S=1:N=1
20  S=S*(1/N)^(2^-N)
30  PRINT S
40  N=N+1
50  GOTO 20
         .
         .
         .
    .6017975
    .6017975
    .6017975
    .6017975
    .6017975
    .6017975
    .6017975
```

It seems that .6017975 is the approximate limit.

(2)(c) This sequence is again one that is defined recursively. The ration s_{n+1}/s_n is greater than one, so the sequence is nondecreasing. If it is bounded above, then it must converge to its least upper bound. Using techniques that we shall develop later in this chapter, it will be easy for us to show that this sequence is bounded above. In any case, the following program plus output tells us that this sequence is bounded above and gives us a good approximation to the limit.

```
10  S=1:N=1
20  S=S*(1+(1/N))^(1/(2^N))
30  PRINT S
40  N=N+1
50  GOTO 20

 1.414213
 1.565085
 1.622389
 1.645175
 1.654575
 1.658565
 1.660296
```

1.66106
1.661402
1.661556
1.661627
1.661659
1.661674
1.661681
1.661685
1.661686
1.661687
1.661687
1.661688
1.661688
1.661688
1.661688

Now Try The Variations

We now begin VARIATIONS on EXERCISE 5.15. Remember that corresponding problems use similar methods. Many of these problems have "general ideas" behind them. By the time you have worked the last variation of each type, you should try to articulate these ideas. Don't worry about being wrong in stating these general ideas, just try to make sense!

5.17 VARIATIONS ON EXERCISE 5.15

(1) Discuss the convergence or divergence of the following sequences:

(a) $\dfrac{4n^5 + 3n^4 + 1}{7n^5 + 2}$, $n = 0, 1, 2, \ldots$

(b) $\dfrac{-n^3 + 1}{2n^4 + 3}$, $n = 0, 1, 2, \ldots$

(c) $\dfrac{(-n)^n + n^{n-1}}{n^n - n^{n-1}}$, $n = 2, \ldots$

(d) $n^n/(1 + n^{-1})^{n^2}$, $n = 1, 2, \ldots$

(e) $\dfrac{(-n)^3 + 1}{n^3 + (\ln(n))^5}$, $n = 0, 1, 2, \ldots$

(f) $\cos\left(\pi \dfrac{n^2 + 1}{n + 2}\right)$, n = 0, 1, 2, . . .

(g) $\dfrac{\log_2(n^5)}{\log_3(n)}$, n = 2, 3, . . .

(h) $\dfrac{\log_2(n)}{n^{0.001}}$, n = 1, 2, . . .

(i) $\dfrac{\log_2(\log_2(n))}{\log_2(n)}$, n = 2, 3, . . .

(j) $\cos\left(\dfrac{n^2 + 1}{n + 2}\right)$, n = 0, 1, 2, . . .

(k) $\left(1 + \dfrac{2}{n}\right)^n$, n = 1, 2, . . .

(2) By using a computer to generate these sequences, discuss their convergence or divergence:

(a) $\sin((44n^3 + n + 1)/(7n^2 + 1))$, n = 0, 1, 2, . . .

(b) s_n, n = 1, 2, . . ., where $s_{n+1} = \left(1 + \dfrac{1}{n}\right)^{(\ln(n))^{-1}} s_n$ for n = 2, 3, . . ., and $s_1 = 1$

(c) s_n, n = 1, 2, . . ., where $s_{n+1} = \left(\dfrac{n + 1}{2n + 1}\right)^{\frac{(-1)^n}{n}} s_n$ and $s_1 = 1$

5.18 VARIATIONS ON EXERCISE 5.15

(1) Discuss the convergence or divergence of the following sequences:

(a) $\dfrac{4(\ln(n))^5 + 3(\ln(n))^4 + 1}{7(\ln(n))^5 + 2}$, n = 1, 2, . . .

(b) $\dfrac{-n^{-3} + 1}{2n^{-4} + 3}$, n = 1, 2, . . .

(c) $\dfrac{(-n)^{n-1} + n^n}{n^n - n^{n-1}}$, n = 2, . . .

(d) $(1 + n)^{1/n}$, n = 1, 2, . . .

(e) $\dfrac{(-n\ln(n))^3 + 1}{n^3 + (\ln(n))^5}$, n = 1, 2, . . .

(f) $\sin\left(\dfrac{\pi^2 n^2 + 1}{\pi n + 2}\right)$, n = 0, 1, 2, . . .

(g) $\dfrac{\log_2(n^5 + \log_2(n))}{\log_3(n)}$, n = 2, 3, . . .

(h) $\dfrac{\ln(n)}{n^{(\ln(n))^{-1}}}$, n = 2, 3, . . .

(i) $\dfrac{\log_2(\log_2(n))}{\log_3(\log_3(n))}$, n = 4, 5, . . .

(j) sin(2n), n = 0, 1, 2, . . .

(k) $\left(1 + \dfrac{\ln(n)}{n}\right)^n$, n = 1, 2, . . .

(2) By using a computer to generate these sequences, discuss their convergence or divergence:

(a) $\cos((n^3 + 1)/(n + 1))$, n = 0, 1, 2, . . .

(b) s_n, n = 1, 2, . . ., where $s_{n+1} = (1/n^p)^{q^{-n}} s_n$ and $s_1 = 1$ for p = 1, 2, 3 and q = 2, 3, 4.

(c) s_n, n = 1, 2, . . ., where $s_{n+1} = \left(\dfrac{\log_2(n)}{\log_2(qn)}\right)^{p^{-n}} s_n$, n = 2, 3, 4, . . ., and $s_1 = 1$ for p = 2, 3, 4 and q = 2, 4, 8.

5.19 VARIATIONS ON EXERCISE 5.15

(1) Discuss the convergence or divergence of the following sequences:

(a) $\dfrac{4a_n^5 + 3a_n^4 + 1}{7a_n^5 + 2}$, n = 0, 1, 2, . . ., where $a_n = (n^2 + 1)/(2n^2 + 1)$

(b) $\dfrac{-\sin^3 n + 1}{2\sin^4 n + 3}$, n = 0, 1, 2, . . .

(c) $\dfrac{(-n)^n + n^n}{n^n - n^{n-1}}$, n = 2, . . .

(d) $(2 + \sin(n))^{1/n}$, n = 1, 2, . . .

(e) $\dfrac{(-n\ln(n))^3 + 2n^3}{n^3 + 2(n\ln(n))^3}$, n = 1, 2, . . .

(f) $\sin\left(\dfrac{\pi^2 n^2 + n}{\pi n + 2}\right)$, n = 0, 1, 2, . . .

(g) $\dfrac{\log_2(n^5 + 2^n)}{n}$, n = 2, 3, . . .

(h) $\dfrac{\ln(n)}{n^{(\ln(\ln(n))^{-1}}}$, n = 4, 5, . . .

(i) $\dfrac{\log_2(\log_2(\log_2(n)))}{\log_3(\log_3(\log_3(n)))}$, n = 28, 29, . . . The general rule?

(j) sin(3n), n = 0, 1, 2, . . . The general rule?

(k) $\left(1 + \dfrac{\ln(n)}{n}\right)^{n/\ln(n)}$, n = 2, 3, . . .

(2) By using a computer to generate these sequences, discuss their convergence or divergence:

(a) s_n, n = 1, 2, 3, . . ., where $s_{n+1} = \frac{(2n)^2}{(2n - 1)(2n + 1)} s_n$, n = 1, 2, 3, . . ., and $s_1 = 2$.

(b) $\sin(s_n n)$, n = 1, 2, 3, . . ., where s_n is as in (a).

(c) $\sin(s_n \lfloor \ln(\ln(n + 3)) \rfloor)$, n = 1, 2, 3, . . ., where s_n is as in (a). The function $\lfloor x \rfloor$ is the "greatest integer function" or "floor function" and is called INT in BASIC. For example, INT(4.7) = 4, INT(3.14) = 3, etc.

An Infinite Series Is A Sequence Of Partial Sums

We are now ready to start our discussion of infinite series. Let's start with an old friend, an infinite *sequence* a_k, k = 0, 1, 2, ... Remember, this is nothing more than a function from the domain {0, 1, 2, . . .} to the real numbers. The value of the function at k is denoted by a_k. If we have another sequence b_k, k = 0, 1, 2, . . ., then we say the two sequences are the same or are "equal" if $a_k = b_k$ for all k, k = 0, 1, 2, ... Suppose that for each integer n, n = 0, 1, 2, . . ., we form the sum $s_n = a_0 + a_1 + \ldots + a_n$. This defines a new sequence s_n, n = 0, 1, 2, . . ., called the *infinite series with terms from the infinite sequence* a_k, k = 0, 1, 2, ... The sequence s_n is also called the *sequence of partial sums* of the sequence a_k, k = 0, 1, 2, ... If s_n, n = 0, 1, 2, . . ., is the sequence of partial sums of a_k, k = 0, 1, 2, . . ., and t_n, n = 0, 1, 2, . . ., is the sequence of partial sums of b_k, k = 0, 1, 2, . . ., then to say the sequence (s_n) equals the sequence (t_n) means that $s_n = t_n$, n = 0, 1, 2, ... In particular, $s_0 = t_0$, so $a_0 = b_0$. In general, for n > 0, $a_n = s_n - s_{n-1} = t_n - t_{n-1} = b_n$, and hence $a_n = b_n$ for all n. Thus, two sequences can give rise to the same sequence of partial sums if and only if these two sequences are themselves equal.

Infinite Sums—Two Interpretations

We now confront another notational artifact of calculus. The infinite series with terms from the infinite sequence a_k, k = 0, 1, 2, . . ., is denoted by

$$\sum_{k=0}^{\infty} a_k .$$

So, nothing wrong with that! Unfortunately, if $s_n = a_0 + a_1 + \ldots + a_n$ is the n^{th} partial sum, we also have the notation

$$\sum_{k=0}^{\infty} a_k = \operatorname{limit}_{n\to\infty} s_n = A$$

which says that the limit of the sequence of partial sums is A. In other words, the same infinite summation notation is used to designate both the sequence and its limit, two very different things. Usually, there is enough additional information floating around in any given discussion to avoid confusion.

Equal Series . . . Equal Limits, It's Not The Same

In general, if you want to say that two infinite series are equal, you say something like

"The infinite series $\sum_{k=0}^{\infty} a_k$ equals the infinite series $\sum_{k=0}^{\infty} b_n$."

This would mean that $a_k = b_k$ for all k. On the other hand, the statement

$$\sum_{k=0}^{\infty} a_k = \sum_{k=0}^{\infty} b_k,$$

by itself usually means that the limits of the two series are equal and does not mean that $a_k = b_k$ for all k.

As a specific example, let $a_k = (1/2)^k$, $k = 0, 1, 2, \ldots$, and let $b_0 = 0$, $b_k = (2/3)^k$, $k = 1, 2, \ldots$. These "geometric series" are studied in high school algebra or precalculus courses and both converge to the number 2. Thus, we write

$$\sum_{k=0}^{\infty} a_k = \sum_{k=0}^{\infty} b_k = 2.$$

Every Sequence Is A Sequence Of Partial Sums

Before studying examples of series, there is one other general remark to be made. It seems from the definition that infinite series are a special class of infinite sequences. This is true in the sense that an infinite series s_n, $n = 1, 2, \ldots$, is specified as the sequence of partial sums of some infinite sequence a_n, $n = 0, 1, 2, \ldots$. Because of this, we discuss infinite series in terms of this underlying sequence a_n and its properties, giving the theory of infinite series a special notational and conceptual flavor. You should realize, however, that any sequence b_n, $n = 0, 1, 2, \ldots$, can be regarded as an infinite series. Just define $a_0 = b_0$ and $a_n = b_n - b_{n-1}$, $n = 1, 2, \ldots$, and the sequence $b_n = s_n$, where $s_{\hat{n}}$, n = 0, 1, 2, . . ., is the sequence of partial sums of a_n, n = 0, 1, 2, ... This is the type of remark that interests mathematicians and bores everyone else. It does come in handy in some specific problems concerning sequences and series, so at least take note of it.

5.20 EXAMPLE: GEOMETRIC SERIES Let $a_k = r^k$, k = 0, 1, 2, . . ., where r is any real number. The infinite series with terms from this sequence is the sequence of partial sums $s_n = 1 + r + \ldots + r^n$, n = 0, 1, 2, ... We know from precalculus courses that $s_n = (1 - r^{n+1})/(1 - r)$ if $r \neq 1$. If r = 1 then $s_n = n + 1$. It is obvious from this formula that the sequence s_n diverges if $|r| \geqslant 1$ and converges to $1/(1 - r)$ if $|r| < 1$. Thus, we write

$$(*) \sum_{k=0}^{\infty} r^k = \frac{1}{1 - r} \text{ if } |r| < 1.$$

As examples of geometric series, take r = (1/2) and r = (2/3) to get

$$\sum_{k=0}^{\infty} (1/2)^k = \frac{1}{1 - (1/2)} = 2 \text{ and } \sum_{k=0}^{\infty} (2/3)^k = \frac{1}{1 - (2/3)} = 3.$$

If the first term is missing from a geometric series (*) then the sum of the series is $r/(1 - r)$ instead of $1/(1 - r)$. Thus, for example,

$$\sum_{k=1}^{\infty} (2/3)^k = \frac{(2/3)}{1 - (2/3)} = 2.$$

Our next example has some very special properties that are important to understanding series.

5.21 EXAMPLE: HARMONIC AND ALTERNATING HARMONIC SERIES Let $a_k = 1/k$, k = 1, 2, ... The series of partial sums $s_n = 1 + (1/2) + (1/3) + \ldots + (1/n)$, n = 1, 2, . . ., is called the *harmonic series*. This series diverges. One way to see this is to write the series as

$$1 + [(1/2)] + [(1/3) + (1/4)] + [(1/5) + (1/6) + (1/7) + (1/8)] + \ldots$$

where the general term in square brackets is

$$[(1/(2^n + 1)) + \ldots + (1/2^{n+1})].$$

Each term in square brackets is greater than or equal to 1/2 and there are infinitely many such terms, thus the series must diverge.

An important related series is the *alternating harmonic series* where $a_k = (-1)^{k-1}(1/k)$, $k = 1, 2, \ldots$. The partial sums of this series are of the form $s_n = 1 - (1/2) + (1/3) - (1/4) + \ldots + (-1)^{n-1}(1/n)$. This series converges. To see why intuitively, imagine that you are standing in a room with your back against a wall. Imagine that you step forward 1 meter, then backward 1/2 meter, then forward 1/3 meter, etc. After n such steps, your distance from the wall is the value of the partial sum s_n. By the time you are stepping forward one millimeter, etc., an observer in the room (who by now has decided you are nuts) would conclude that you are standing still. In other words, you have converged. This argument works just as well for any step sizes, as long as they are alternating forward and backward, of decreasing size, and tend to zero. In the case of the alternating harmonic series, your final distance from the wall is ln(2) meters. Check it out on your computer. See also EXERCISE 5.70(16).

Alternating Series

From the discussion of EXAMPLE 5.21, we have the following theorem:

5.22 THEOREM Let a_k, $k = 1, 2, \ldots$, be a sequence of positive numbers such that $a_k \geqslant a_{k+1}$ for $k = 1, 2, \ldots$. If $\lim_{k\to\infty} a_k = 0$ then

$$\sum_{k=1}^{\infty} (-1)^{k-1} a_k$$

converges and

$$\left| \sum_{k=1}^{\infty} (-1)^{k-1} a_k - \sum_{k=1}^{n} (-1)^{k-1} a_k \right| \leqslant |a_{n+1}|$$

Proof: The intuitive idea of the proof was discussed in connection with EXAMPLE 5.21. Let s_{2n}, $n = 1, 2, \ldots$, be the sequence of even partial sums. We write

$$s_{2n} = (a_1 - a_2) + (a_3 - a_4) + \ldots + (a_{2n-1} - a_{2n}).$$

Each of the terms in parenthesis is nonnegative. Thus the sequence s_{2n} is nondecreasing. The sequence s_{2n} is also bounded above by a_1. By THEOREM 5.13, this sequence converges to its least upper bound, LU. Similarly, define the sequence s_{2n+1}, $n = 0, 1, \ldots$, of odd partial sums. This sequence can be written

$$s_{2n+1} = a_1 - (a_2 - a_3) - (a_4 - a_5) - \ldots - (a_{2n} - a_{2n+1}).$$

This sequence is obviously nonincreasing and bounded below by zero. Thus, it converges to its greatest lower bound, GL. But, $|s_{2n+1} - s_{2n}| = a_{2n+1}$ tends to zero as n tends to infinity and hence GL = LU is the limit of the sequence s_n, n = 1, 2, 3, ... This proves that the series converges. The ideas of EXAMPLE 5.21 explain the error estimate.

Absolute Convergence

There is the germ of another important idea in EXAMPLE 5.21.

5.23 DEFINITION Let s_n, n = 0, 1, 2, . . ., be the series with terms a_k, k = 0, 1, 2, ... Let t_n be the series with terms $|a_k|$, k = 0, 1, 2, ... If the series t_n converges, then the series a_n is said to *converge absolutely* or to be an *absolutely convergent series*

Another way that DEFINITION 5.23 is stated is

$$\sum_{k=0}^{\infty} a_k \text{ converges absolutely if } \sum_{k=0}^{\infty} |a_k| \text{ converges.}$$

It may not be obvious to you that a series that converges absolutely must converge. We shall explain why below. The converse is false, in that a series that converges need not converge absolutely. This is the case with the alternating harmonic series, which converges but does not converge absolutely because the harmonic series diverges.

Convergent Series—A Thought Algorithm

It's time to refresh our intuition about infinite series. Remember, we start with a sequence a_k, k = 0, 1, 2, ... These are the terms of the infinite series s_n, n = 0, 1, 2, . . ., where $s_n = a_0 + a_1 + \cdots + a_n$. Imagine that we have written a computer program to evaluate these numbers s_n. We start the computer program running and watch the screen. The numbers s_n stream onto the screen. At first we are thrilled. The numbers are changing rapidly. But after awhile they begin to change only in the 10^{th} decimal place and we become bored. This is the sort of thing that happens when the series is converging to some number A. We may never see A on the screen, only numbers close to A. Here is a definition of convergence of infinite series that corresponds to our computer intuition.

Alternative Definition Of Convergence

5.24 DEFINITION Let s_n, n = 0, 1, 2, . . ., be the sequence of partial sums of the sequence a_k, k = 0, 1, 2, ... The sequence s_n converges if for every $\epsilon > 0$ there exists an integer N_ϵ such that for all $q \geq p > N_\epsilon$, $|a_p + \ldots + a_q| < \epsilon$.

In our imaginary computer experiment, the ϵ is a small number, like 10^{-9}, where, if the numbers are changing by an amount less than that number, we lose interest. The number N_ϵ corresponds to the number of terms in the sum that have been computed when this begins to happen. After this begins to happen, we can watch the computer add any amount of additional terms from any p to any q and the sum won't change by much. The advantage of DEFINITION 5.24 is that it doesn't explicitly mention the limit of the series. Of course, we have already given the definition of the convergence of a series in terms of the convergence of the sequence of partial sums. This uses DEFINITION 5.7. Technically this means that DEFINITION 5.24 is really a little theorem. We won't worry about this technicality.

Absolute Convergence Implies Convergence

Just to give DEFINITION 5.24 a try, let's prove that a series that converges absolutely must converge.

5.25 THEOREM If a series converges absolutely then it converges.

Proof: Let t_n, n = 0, 1, 2, . . ., be the sequence of partial sums of the sequence $|a_0|, |a_1|, |a_2|, \ldots$. We assume that t_n converges. Thus, given any $\epsilon > 0$, there exists an N_ϵ such that for $q \geq p > N_\epsilon$, $|a_p| + \ldots + |a_q| < \epsilon$. But, $|a_p + \ldots + a_q| \leq |a_p| + \ldots + |a_q|$ and hence, for $q \geq p > N_\epsilon$, $|a_p + \ldots + a_q| < \epsilon$. By DEFINITION 5.24, this proves that the sequence s_n of partial sums of a_k, k = 0, 1, 2, . . ., converges. This completes the proof.

Alternating Series

5.26 DEFINITION Let a_k, k = 0, 1, 2, . . ., be an infinite sequence. For any integer t = 0, 1, 2, . . ., we consider the sequence $a_t, a_{t+1}, a_{t+2}, \ldots$. This sequence will be called the "t^{th} tail sequence of the sequence a_k." The sequence of partial sums $s_n^t = a_t + a_{t+1} + \ldots + a_{t+n}$, n = 0, 1, 2,

. . ., is the "t^{th} tail series" of the series $s_n = a_0 + \ldots + a_n$, n = 0, 1, 2, ...

In the infinite sum notation, we say that

$$\text{"the series } \sum_{k=t}^{\infty} a_k \text{ is the } t^{th} \text{ tail of the series } \sum_{k=0}^{\infty} a_k\text{"}$$

In this context, we are using the infinite sum notation to specify the series or sequence of partial sums and not the limit. When we say

$$\sum_{k=0}^{\infty} a_k = a_0 + a_1 + \ldots + a_{t-1} + \sum_{k=t}^{\infty} a_k$$

we are using the infinite sum notation to denote the limits of these respective series. Again, although this notation is not the greatest, it is concise and doesn't usually lead to confusion.

It's Enough To Test The Tail . . .

The next result is simple, but extremely useful for testing series for convergence.

5.27 THEOREM If any tail series of a series converges then the series converges. Conversely, if any tail series diverges then the series diverges.

Proof: Suppose that a_k, k = 0, 1, 2, . . ., are the terms of the series. To say that the t^{th} tail series converges means that given any $\epsilon > 0$ there exists N_ϵ such that for all $q \geqslant p > N_\epsilon$, $|a_p + a_{p+1} + \ldots + a_q| < \epsilon$. This is exactly what is required by DEFINITION 5.24 for the whole series to converge. Conversely, if the tail series diverges then there is some $\epsilon > 0$ such that for every integer N there exists $q \geqslant p > N$ such that $|a_p + \ldots + a_q| \geqslant \epsilon$. This shows that the conditions of DEFINITION 5.24 are not valid when applied to the whole series and thus the whole series diverges.

As an example of the way THEOREM 5.27 is used, consider the series with terms $a_k = (-1)^k 100/|k - 99.5|$, k = 1, 2, ... We would like to apply THEOREM 5.22, but the hypothesis of this theorem doesn't quite apply. In particular, we don't have $|a_k| \geqslant |a_{k+1}|$ for k = 1, 2, ... The sequence $|a_k|$ is increasing until k = 99, then $|a_{99}| = |a_{100}| = 200$. After that, the sequence $|a_k|$ does satisfy the conditions of THEOREM 5.22, and hence the

t^{th} tail series converges for $t = 100$. By THEOREM 5.27, the whole series converges. We are using $|a_k|$ for what we called a_k in THEOREM 5.22.

Here is another useful result that follows from DEFINITION 5.24.

A Result About Products Of Sequences

5.28 THEOREM Suppose that the series with terms a_k, $k = 0, 1, 2, \ldots$, converges absolutely and let b_k, $k = 0, 1, 2, \ldots$, be any bounded sequence. Then the series with terms $a_k b_k$, $k = 0, 1, 2, \ldots$, converges absolutely.

Proof: Let M be such that $|b_k| < M$ for all k. By hypothesis, the series $|a_0| + |a_1| + \ldots + |a_n| + \ldots$, converges. Given any $\epsilon > 0$, there exists an N such that for all $q \geq p > N$, $|a_q| + \ldots + |a_p| < \epsilon/M$. This implies that $|a_q b_q| + \ldots + |a_p b_p| < \epsilon$. Thus, by DEFINITION 5.24, the series with terms $a_k b_k$ converges absolutely. This completes the proof.

We'll learn later that the series with terms $a_k = 1/k^p$, $k = 1, 2, 3, \ldots$, converges if $p > 1$ and the convergence is obviously absolute since the terms are all positive. Thus, if we take $p = 3$, then the series $1 + 2^{-3} + 3^{-3} + 4^{-3} + \ldots$ converges absolutely. If we multiply the terms of this series by the divergent, unbounded sequence $b_k = k$, then we still obtain an absolutely convergent series, $1 + 2^{-2} + 3^{-2} + 4^{-2} + \ldots$. This shows that the converse of THEOREM 5.28 is false in that absolute convergence of the series with terms $a_k b_k$ does not imply that the sequence b_k is bounded.

The Comparison Test For Convergence

One of the easiest ways to determine the absolute convergence of a series is to compare it with another series known to converge absolutely. The idea is stated in the following corollary.

5.29 COROLLARY (COMPARISON TEST) Suppose that the series with terms a_k, $k = 0, 1, 2, \ldots$, converges absolutely. Let c_k, $k = 0, 1, 2, \ldots$, be a sequence such that $|c_k| \leq M|a_k|$, $k = 0, 1, 2, \ldots$, where M is some positive real number. Then, the series with terms c_k, $k = 0, 1, 2, \ldots$, converges absolutely.

Proof: We apply THEOREM 5.28. Define a sequence b_k by $b_k = 0$ if $a_k = 0$ and, otherwise, $b_k = c_k/a_k$. This sequence c_k is bounded by M. Thus, by THEOREM 5.28, the sequence $a_k b_k = c_k$ converges absolutely. This completes the proof.

The Comparison Test For Divergence

Another way to state the COMPARISON TEST is that if the series with terms $|a_k|$, $k = 0, 1, 2, \ldots$, diverges and if $C|c_k| \geqslant |a_k|$, for $k = 0, 1, 2, \ldots$ and $C > 0$, then the series with terms $|c_k|$, $k = 0, 1, 2, \ldots$, diverges. To see this, we note that if the series with terms $|c_k|$ converged then, by setting $M = C^{-1}$ in COROLLARY 5.29, we would have that, contrary to assumption, the series with terms $|a_k|$ would converge. In this application of COROLLARY 5.29, the roles of the a_k and c_k are reversed.

By now you are thinking "ENOUGH THEORY!" You are right. It's time to put this stuff to work. One of the best ways to learn a subject is to teach it. Our goal in what follows is to show you how to be a good instructor in the subject of infinite series, by teaching you how to make up problems for your classmates to solve.

Now You Learn To Make Up Infinite Series Problems

5.30 PROBLEMS BASED ON THE INTEGRAL TEST Suppose that we have a continuous function $f(x)$ such that $f(x) > 0$ and $f(x)$ is nonincreasing for $x > t > 0$. Let a_k, $k = 0, 1, 2, \ldots$, be a sequence such that $|a_k| \leqslant f(k)$, $k \geqslant t$. Then the "integral test" states

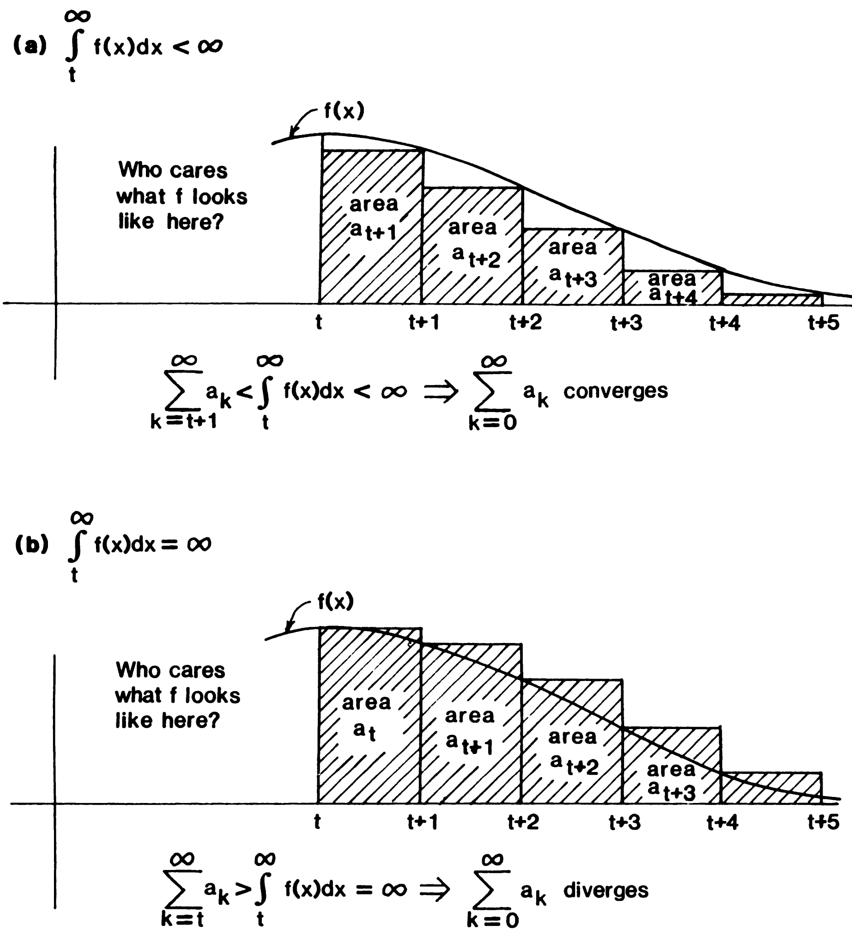
$$\textstyle\sum_{k=0}^{\infty} a_k \text{ converges absolutely if } \int_t^{\infty} f(x)dx < \infty.$$

On the other hand, if $|a_k| \geqslant f(k)$ for $k \geqslant t$ then

$$\sum_{k=0}^{\infty} |a_k| \text{ diverges if } \int_t^{\infty} f(x)dx = \infty.$$

The reason that these results are true is shown in FIGURE 5.31. There we see that the area under the curve is related to the sum of the t^{th} tail of the series with terms a_k in such a way that both either converge or diverge.

FIGURE 5.31 The Integral Test



Look For Positive Nonincreasing Functions

To construct problems based on the integral test, all we need to do is find functions f(x) that eventually, past some real number t, become positive and nonincreasing. We then evaluate the integral from t to infinity of f(x). If it diverges we can construct a divergent series, if it converges we can construct a convergent series.

For example, let's take $f(x) = x^{-1}$. The integral

$$\int \frac{dx}{x} = \ln(x) \text{ so } \int_1^\infty \frac{dx}{x} = \lim_{x \to \infty} \ln(x) - 0 = +\infty.$$

Thus, the series

$$\sum_{k=1}^{\infty} \frac{1}{k}$$

diverges to infinity. We already knew this.

Let's try a function $f(x) = (x\ln(x))^{-1}$ that goes to zero a little bit faster than x^{-1}. Perhaps the related series will converge. We compute

$$\int \frac{dx}{x\ln(x)} = \ln(\ln(x)) \text{ so } \int_2^{\infty} \frac{dx}{x\ln(x)} = \lim_{x\to\infty} \ln(\ln(x)) - \ln(\ln(2)) = +\infty$$

Thus, the series

$$\sum_{k=2}^{\infty} \frac{1}{k\ln(k)} = +\infty$$

is a divergent series also.

Trying once again along these same lines, we discover that

$$\int \frac{dx}{x(\ln(x))^2} = \frac{-1}{\ln(x)} \text{ so } \int_2^{\infty} \frac{dx}{x(\ln(x))^2} < \infty.$$

Thus we obtain the interesting fact that

$$\sum_{k=2}^{\infty} \frac{1}{k(\ln(k))^2}$$

converges.

Some Tricks For Making Difficult Problems

In EXERCISE 5.32, you will be asked to construct four problems based on the integral test and exchange these problems with your classmates. If you want to make these problems appear more difficult than they really are, here is a useful trick. If you have a series with terms a_k that converges absolutely and $\tilde{a}_k$ is any sequence such that the sequence $b_k = \tilde{a}_k/a_k$ is a bounded sequence, then the series with terms $\tilde{a}_k$ also converges absolutely. This is just a restatement of THEOREM 5.28. For example, take

$$a_k = \frac{1}{k(\ln(k))^2} \text{ and } \tilde{a}_k = \frac{3k + 1}{(k^2 + 1)(\ln(k))^2}, \; k = 2, 3, \ldots.$$

It is easy to check that $\lim_{k\to\infty} (\tilde{a}_k/a_k) = 3$ and hence the sequence $b_k = \tilde{a}_k/a_k$, $k = 2, 3, \ldots$, is bounded. Thus, the series

$$\sum_{k=2}^{\infty} \frac{3k + 1}{(k^2 + 1)(\ln(k))^2}$$

converges. To make this look even more messy, the expression ln(k) could be replaced by someting like $\ln((k^2 + 1)/(k + 2))$. The same sort of trick can be played with divergent sequences.

Check The Tables And Chapter 4 For Integrals

In finding integrals to use in constructing convergent and divergent series, don't forget to browse through the problems from CHAPTER 4 and the TABLE OF INTEGRALS in the back of the book. The TABLE OF INTEGRALS contains a section called MISCELLANEOUS DEFINITE INTEGRALS that has some interesting integrals. Two such integrals are

$$\int_0^{\infty} \ln\left(\frac{e^x + 1}{e^x - 1}\right) dx \text{ and } \int_0^1 \ln\left(\frac{1 + x}{1 - x}\right) \frac{dx}{x}.$$

The second integral can be transformed into the first by the substitution $x = e^{-v}$. Using the first integrand, we compute

$$\frac{d}{dx} \ln\left(\frac{e^x + 1}{e^x - 1}\right) = \frac{-2e^x}{e^{2x} - 1}.$$

Thus the function $\ln((e^x + 1)/(e^x - 1)$ is positive and decreasing for $x > 0$. According to our integral table, the integral of this function from 0 to infinity is $\pi^2/4$. In particular, the integral from 1 to infinity is finite and hence

$$\sum_{k=1}^{\infty} \ln\left(\frac{e^k + 1}{e^k - 1}\right)$$

converges.

The integral test, as we have used it thus far, is based on integrals where the upper limit is infinity. The second integral mentioned above

$$\int_0^1 \ln\left(\frac{1 + x}{1 - x}\right) \frac{dx}{x}$$

has upper limit 1. As remarked above, by making the change of variable $x = e^{-v}$, this integral can be transformed into the integral of the previous paragraph. Instead, let's make the change of variable $x = v^{-1}$. Then $dx = -v^{-2}dv$ and the integral becomes

$$-\int_{\infty}^{0} \ln\left(\frac{1 + v^{-1}}{1 - v^{-1}}\right) \frac{dv}{v} = \int_0^{\infty} \ln\left(\frac{v + 1}{v - 1}\right) \frac{dv}{v}.$$

This means that the series

$$\sum_{k=2}^{\infty} \frac{1}{k} \ln\left(\frac{k+1}{k-1}\right)$$

converges.

As one final example of the integral test, consider the integral

$$\int x^{-p}dx = \frac{x^{-p+1}}{-p+1} \text{ if } p > 1 \text{ so } \int_1^{\infty} x^{-p}dx = \frac{1}{p-1} \text{ if } p > 1.$$

This means that the series

$$\sum_{k=1}^{\infty} k^{-p}$$

converges if $p > 1$. Otherwise, by the integral test again, the series diverges.

Transforming Integrals Into Series

So now it's your turn to make up some exercises for your classmates based on the integral test. BE MEAN!

5.32 EXERCISES Make up four exercises based on the integral test to exchange with your classmates.

5.33 PROBLEMS BASED ON THE COMPARISON TEST Suppose the infinite series

$$\sum_{k=0}^{\infty} a_k$$

converges absolutely. COMPARISON TEST 5.29 says that if c_k, $k = 0, 1, 2, \ldots$, is a sequence such that $|c_k| \leq M|a_k|$ for some positive real number M and $k = 0, 1, 2, \ldots$, then the series

$$\sum_{k=0}^{\infty} c_k$$

also converges absolutely. The usual way that the inequality $|c_k| \leq M|a_k|$ is established is by showing that

$$\lim_{k \to \infty} \frac{|c_k|}{|a_k|}$$

exists. For example, the rational function

$$c_k = \frac{3k + 3}{k^2 + 5} = \frac{3 + (3/k)}{k + (5/k)}$$

is very close to $a_k = 3/k$ for large values of k. In fact, the ratio c_k/a_k approaches 1 as k tends to infinity. The series with terms 3/k diverges and thus so does the series with terms c_k.

On the other hand, the series with terms c_k defined by

$$c_k = \frac{3k + 3}{k^3 + 5} = \frac{3 + (3/k)}{k^2 + (5/k)}$$

has terms that look like $3/k^2$ for large values of k. The series with terms $3/k^2$ converges and therefore so does the series

$$\sum_{k=0}^{\infty} \frac{3k + 3}{k^3 + 5} .$$

Thus, if you know that a series $a_0 + a_1 + \ldots$ converges absolutely, and you define a series c_k, $k = 0, 1, 2, \ldots$, such that the limit $|c_k/a_k|$ exists, then the series $c_0 + c_1 + \ldots$ converges absolutely.

Two Series Behave The Same If Their Terms Behave The Same

5.34 EXERCISES Make up four exercises based on the comparison test to exchange with your classmates.

5.35 PROBLEMS BASED ON THE ROOT TEST The method called the ROOT TEST for convergence is a special case of the comparison test just discussed. We take $a_k = r^k$ so that the series $a_0 + a_1 + \ldots$ is the geometric series. If b_k, $k = 0, 1, 2, \ldots$, is any sequence with $\lim_{k\to\infty} |b_k| < 1$, then there is some $r < 1$ and some integer t such that, for all $k \geq t$, $|b_k| < r$. Thus $|b_k|^k < r^k$ for $k \geq t$, and hence

$$\sum_{k=t}^{\infty} |b_k|^k \text{ and thus } \sum_{k=0}^{\infty} |b_k|^k$$

converges. For example, take

$$b_k = \frac{2k^2 + k + 5}{3k^2 - k + 6} .$$

The limit of the sequence b_k is 2/3. It then follows that the series

$$\sum_{k=0}^{\infty} \left(\frac{2k^2 + k + 5}{3k^2 - k + 6}\right)^k$$

converges.

Another example is gotten by taking $b_k = (1 - (1/k))^k$, $k = 1, 2, \ldots$. The limit of b_k as k tends to infinity is e^{-1}. Thus, the series

$$\sum_{k=1}^{\infty} (1 - (1/k))^{k^2}$$

converges. If the limit as k tends to infinity of $|b_k|$ is greater than 1, then the sequence $|b_k|^k$ doesn't tend to zero, and consequently

$$\sum_{k=0}^{\infty} |b_k|^k$$

diverges. If the limit of $|b_k|$ is equal to 1, then the series

$$\sum_{k=0}^{\infty} |b_k|^k$$

can either converge or diverge. For example, $b_k = k^{(-1/k)}$ and $b_k = k^{(-2/k)}$ both tend to 1 as k tends to infinity. In the first case, the series

$$\sum_{k=0}^{\infty} |b_k|^k$$

diverges and in the second case it converges.

Root Test: Let limit $|b_k| = \beta$; $\Sigma\, |b_k|^k$ Converges If $\beta < 1$, Diverges If $\beta > 1$

There is a very common class of series whose convergence or divergence can be determined by the root test. These series, which look sort of like the geometric series, have the following form:

$$\sum_{k=0}^{\infty} f(k)r^k, \text{ r a real number, } f(k) > 0.$$

If we think of $f(k)r^k = |b_k|^k$, then $|b_k| = (f(k))^{1/k}r$. In most examples, f(k) is chosen such that the limit as k goes to infinity of $f(k)^{1/k}$ is 1. Here are some examples of such f(k): $f(k) = k$, $f(k) = k^2$, $f(k) = k^p$, p any real number, $f(k) = 3k^3 + 2k^2 + k + 1$, f(k) any positive valued polynomial in k, f(k) any positive valued rational function of k. In most cases, the way you show that $f(k)^{1/k}$ tends to 1 is by showing that its logarithm, $(1/k)\ln(f(k))$ tends to zero as k goes to infinity. Here is a series that converges, constructed by this method

$$\sum_{k=0}^{\infty} \left(\frac{9k^5 + 5k}{k^2 + 3}\right)^3 (1/2)^k.$$

In this series, the (1/2) can be replaced by any function of k that has limit 1/2. For example, (2k + 1)/(4k + 3) could be used in place of the 1/2.

The name ROOT TEST that is given to this method comes from the fact that the limit condition is put on the numbers $|b_k|$, which are the k^{th} roots of the terms $|b_k|^k$ that appear in the series. This method is also called CAUCHY'S TEST. If the series is initially written $\sum_{k=0}^{\infty} a_k$, then the root test is applied to $|b_k| = |a_k|^{1/k}$. The series converges absolutely if α = limit $|a_k|^{1/k} < 1$, diverges absolutely if $\alpha > 1$.

Root Test: Let limit $|a_k|^{1/k} = \alpha$; $\Sigma |a_k|$ Converges If $\alpha < 1$, Diverges If $\alpha > 1$

5.36 EXERCISES Make up four exercises based on the root test to exchange with your classmates.

5.37 PROBLEMS BASED ON THE RATIO TEST The method we now discuss is called the RATIO TEST for convergence. This method is again based on comparison with the geometric series. It is a less powerful method than the root test discussed previously. If we have a series with terms a_k, k = 0, 1, 2, . . ., then we may consider

$$\lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|} = \rho.$$

The RATIO TEST states that if $\rho < 1$ then the series converges absolutely, and if $\rho > 1$ then the series $|a_k|$, k = 0, 1, 2, . . ., diverges. It is not too difficult to show that if the limit of $|a_{k+1}|/|a_k|$ is ρ then the limit of $|a_k|^{1/k}$ is also ρ. The converse is not true. For example, if we take

$$a_k = 2^{-k-(-1)^k}$$

then

$$\lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|} = \lim_{k\to\infty} 2^{-1+2(-1)^k}$$

which does not exist. On the other hand,

$$\lim_{k\to\infty} |a_k|^{1/k} = \lim_{k\to\infty} 2^{-1-(-1)^k/k} = 1/2$$

shows that the series converges.

The Ratio Test Is Less Powerful But Sometimes More Useful

The ratio test is sometimes more useful than the root test in dealing with series whose terms involve factorials. Consider the series

$$\sum_{k=1}^{\infty} \frac{(k!)^2}{(2k)!}.$$

For this series, we obtain

$$\lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|} = \lim_{k\to\infty} \frac{(k+1)^2}{(2k+1)(2k+2)} = 1/4.$$

Thus the series converges. Applying the root test involves taking the limit of $(k!)^{2/k}$ and $((2k)!)^{1/k}$. The best way to do this is to use a formula called STIRLING'S FORMULA, which approximates k! by $(2\pi k)^{1/2}(k/e)^k$. Making this substitution for k! and the corresponding substitution of $(2\pi(2k))^{1/2}(2k/e)^{2k}$ for (2k)! in $a_k = (k!)^2/(2k)!$, taking the k^{th} root, and then taking the limit also gives 1/4. The ratio test is much easier in this example!

It is easy to see why the ratio test is valid for determining convergence. Suppose that the limit of $|a_{k+1}|/|a_k|$ is less than 1. Then there is some positive $r < 1$ and some integer t, such that for all $k > t$, $|a_{k+1}|/|a_k| < r$. In particular, $|a_{t+1}| < r|a_t|$, $|a_{t+2}| < r^2|a_t|$, . . ., $|a_{t+k}| < r^k|a_t|$, ... The series with terms $|a_t|r^k$, $k = 1, 2, \ldots$, is the geometric series and converges. Thus

$$\sum_{k=t}^{\infty} a_k \text{ converges and hence } \sum_{k=0}^{\infty} a_k \text{ converges.}$$

Ratio Test: Let limit ($|a_{k+1}|/|a_k|$) = α; $\Sigma\, |a_k|$ Converges If $\alpha < 1$, Diverges If $\alpha > 1$

5.38 EXERCISES Work as many of EXERCISE 5.36 as you can using the ratio test.

Conditional Convergence—You Can Get Anything You Want

We have already studied, in EXAMPLE 5.21, the alternating harmonic series

$$\sum_{k=1}^{\infty} (-1)^{k-1}(1/k).$$

This series does not converge absolutely as the harmonic series 1 + (1/2) + (1/3) + . . . diverges. In general, we have the following definition:

5.39 DEFINITION A series that converges but does not converge absolutely is called a *conditionally convergent* series.

Here are some things to be aware of in connection with conditionally convergent series. If we take the conditionally convergent series 1 − (1/2) + (1/3) − . . . and extract the subseries consisting of every other term, we obtain the divergent series 1 + (1/3) + (1/5) . . .. For absolutely convergent series, any subseries also converges. If that doesn't seem very interesting to you, a related fact, pointed out by the mathematician Riemann, states that, given any number, by rearranging the terms of a conditionally convergent series you can make the resulting series converge to exactly that number. We won't have any use for Riemann's result, but it's fun to contemplate and can be proved in a way understandable to the beginner. If, on the other hand, you rearrange the terms of an absolutely convergent series, the new series converges to the same value as the old series.

Another thing you can do to create a new series from a given series $a_0 + a_1 + \ldots$ is to insert parentheses: $(a_0 + a_1) + (a_2 + a_3 + a_4) + (a_5 + a_6) + \ldots$, for example. If the series $a_0 + a_1 + \ldots$ converges, conditionally or absolutely, it doesn't matter, then this new series converges to the same thing. If all of the a_k are nonnegative, then the divergence of the series a_k implies the divergence of the new parenthesized series. For divergent series with both positive and negative terms, be careful. For example, the series 1 − 1 + 1 − 1 . . . is divergent, but (1 − 1) + (1 − 1) + . . . obviously converges to zero.

Conditional Convergence—Dirichlet's Test

5.40 PROBLEMS BASED ON DIRICHLET'S TEST AND ABEL'S TEST Let a_k, k = 0, 1, 2, . . ., be a sequence of nonnegative numbers such that $a_{k+1} \leq a_k$, k = 0, 1, 2, . . ., and the limit of the sequence a_k is zero. In other words, a_k is a nonincreasing sequence which tends to zero. Let b_k, $k = 0, 1, 2, \ldots$, be a sequence with the property that the corresponding sequence of partial sums, $s_n = b_0 + \ldots + b_n$, $n = 0, 1, 2, \ldots$, is bounded. Then the series

$$\sum_{k=0}^{\infty} a_k b_k$$

converges. This result is called DIRICHLET'S TEST. Of course, by replacing a_k by $-a_k$, we see that the result is valid if the a_k are nondecreasing and tend to zero.

THEOREM 5.22 was an example of a class of series of this type. If we take $b_k = (-1)^k$ then clearly the partial sums of this sequence are bounded. This gives the class of series in THEOREM 5.22. In constructing convergent series by the Dirichlet method, the challenge is to think up interesting sequences, b_k, $k = 0, 1, 2, \ldots$, with bounded partial sums. You can also simply make up sequences with bounded partial sums. For example $+1$, $+2$, $+3$, -1, -2, -3, $+1$, $+2$, $+3$, -1, -2, -3, . . . has its partial sums bounded. In the paragraph just prior to EXAMPLE 5.20, we pointed out that every sequence is the sequence of partial sums of some series. Thus, if we start with any bounded sequence c_k, $k = 0, 1, 2, \ldots$, and form the sequence $b_0 = c_0$, $b_k = c_k - c_{k-1}$, $k = 1, 2, \ldots$, then the sequence b_k is a sequence with bounded partial sums, $s_n = c_n$.

Another class of interesting examples can be gotten by taking $b_k = \sin(k\tau + \gamma)$, $k = 0, 1, 2, \ldots$, where γ is any real number and τ is any real number that is not an integral (including zero) multiple of 2π. That the partial sums $b_0 + b_1 + \ldots + b_n$, $n = 0, 1, 2, \ldots$, form a bounded sequence is an easy result in complex analysis. Of course, the sequence $\cos(k\tau + \gamma)$, $k = 0, 1, 2, \ldots$, also has bounded partial sums. Feel free to use these sequences in the EXERCISES 5.41. This means that the following series converge conditionally (remember EXERCISE 5.15 (1-j)).

$$\sum_{k=2}^{\infty} \sin(k)/\ln(k) \qquad \sum_{k=1}^{\infty} \cos(\pi k + 1)/k \qquad \sum_{k=0}^{\infty} \sin(\pi k)/k^{1/2}.$$

Abel's Test

There is a useful variation on Dirichlet's method, called ABEL'S TEST, where instead of just having the sequence of partial sums of the sequence bounded, we make the stronger assumption that the series $b_0 + b_1 + b_2 + \ldots$ actually converges. In this case we can take the sequence a_k, $k = 0, 1,$

2, . . ., to be any nonincreasing sequence or nondecreasing sequence which has a limit. In other words, this "monotonic" sequence tends to a limit, but the limit need not be zero as in DIRICHLET'S TEST. For example, $a_k = 1 + 1/k$, $k = 1, 2, 3, \ldots$, would be such a sequence. ABEL'S TEST states that under these conditions, the series $a_0b_0 + a_1b_1 + \ldots$ converges. In the general statement of Dirichlet's method or Abel's method, we take $k = 0, 1, 2, \ldots$, but the sequences a_k and b_k can start at any value of k and these results are still valid. Remember, the fact that "the tail series of the series converges implies that the series converges" is a result for all convergent series, conditional or absolute. As an application of Abel's method, the series

$$\sum_{k=2}^{\infty} (-1)^k(1 + 1/k)(1/\ln(k))$$

converges.

Both DIRICHLET'S TEST and ABEL'S TEST deal with series formed by taking term-by-term products of sequences where various conditions were put on the sequences. In the case of absolute convergence, we had available the very powerful THEOREM 5.28 which said that the termwise product of a bounded sequence and an absolutely convergent series gave rise to another absolutely convergent series. This theorem is clearly false for conditionally convergent series. As an example, multiply the alternating harmonic series

$$\sum_{k=1}^{\infty} (-1)^{k-1}(1/k)$$

termwise by the bounded sequence $(-1)^{k-1}$, $k = 1, 2, \ldots$.

Dirichlet's And Abel's Test—You Make Up The Problems

5.41 EXERCISES Make up four exercises, using either DIRICHLET'S TEST or ABEL'S TEST, to exchange with your classmates.

Basic Arithmetic Rules For Series

We now give some EXERCISES on infinite series, followed by the SOLUTIONS and VARIATIONS on these exercises. As usual, after reading the solution to a problem, change the original problem slightly and rework it. Then go on to the variations. One thing to keep in mind above all else is the basic fact that an infinite series is a sequence of partial sums. You have had some good solid practice with sequences to fall back on when stuck! In particular, RULES FOR LIMITS OF SEQUENCES 5.10 apply. You should translate these rules into series notation. For example, if $a_0 + a_1 + \ldots + a_k + \ldots$ and $b_0 + b_1 + \ldots + b_k + \ldots$ are two convergent series (conditional or absolute, it makes no difference) and α is any real number, then we have

(1) $$\sum_{k=0}^{\infty} \alpha a_k = \alpha \sum_{k=0}^{\infty} a_k$$

(2) $$\sum_{k=0}^{\infty} (a_k + b_k) = \sum_{k=0}^{\infty} a_k + \sum_{k=0}^{\infty} b_k.$$

How would you define multiplication of two series in terms of their partial sums? Give an example.

Exercises On Series

5.42 EXERCISES

(1) Apply the root test, the ratio test, and the comparison test to each of the following series:

(a) $\displaystyle\sum_{k=1}^{\infty} \frac{(k+1)(k+2)(k+3)}{k!}$

(b) $\displaystyle\sum_{k=1}^{\infty} \frac{2^{k/2}}{k^2 + k + 1}$

(c) $\displaystyle\sum_{k=1}^{\infty} \frac{k+1}{2k+3}$

(d) $\displaystyle\sum_{k=0}^{\infty} \frac{k^5}{5^k}$

(2) Apply the comparison test to each of the following series:

(a) $\displaystyle\sum_{k=0}^{\infty} \frac{1}{k^2 - 150}$

(b) $\displaystyle\sum_{k=5}^{\infty} \frac{1}{2^k - k^2}$

(c) $\displaystyle\sum_{k=2}^{\infty} \frac{1}{(k^3 - k^2 - 1)^{1/2}}$

(d) $\displaystyle\sum_{k=1}^{\infty} \frac{(k+1)^{1/2} - (k-1)^{1/2}}{k}$

(3) Discuss the convergence of the following series:

(a) $\sum_{k=1}^{\infty} \frac{(-1)^k}{k}(1 + 2^{-2} + 3^{-2} + \ldots + k^{-2})$

(b) $\sum_{k=1}^{\infty} \frac{(-1)^k}{k}\left(\frac{1}{1} + \frac{1}{2} + \ldots + \frac{1}{k}\right)$

(4) The geometric series is the series $1 + r + r^2 + \ldots + r^k + \ldots$ where r is any real number. The terms of this series are of the form $r^{f(k)}$ where $f(k) = k$. Discuss the following generalization of the geometric series in which $f(k) = k$ is replaced by $f(k) = (\ln(k))^{\beta}$, β a real number:

$$\sum_{k=1}^{\infty} r^{(\ln(k))^{\beta}}.$$

(5) Discuss whether or not the following series converge or diverge. If convergent, specify whether the series is conditionally convergent or absolutely convergent and explain why.

(a) $\sum_{k=0}^{\infty} (-1)^k \text{arccot}(k)$

(b) $\sum_{k=0}^{\infty} \frac{\sin(k)}{|k - 99.5|}$

(c) $\sum_{k=1}^{\infty} \frac{(-1)^k}{k}\cos(1/k)$

(d) $\sum_{k=0}^{\infty} \frac{-9k^2 - 5}{k^3 + 1}\sin(k)$

Study The Solution—Change And Rework The Problem

5.43 SOLUTIONS TO EXERCISE 5.42

(1)(a) We are asked to apply three methods, the root test, the ratio test, and the comparison test to this problem. We'll do all three methods, but first let's think a bit. If a_k denotes the k^{th} term of this series, then the numerators of a_k is a polynomial of degree 3 and the denominator is an expression that grows faster than any polynomial in k. If the denominator were only the degree 5 polynomial $p(k) = k(k - 1)(k - 2)(k - 3)(k - 4)$, then by comparison with the series with terms k^{-2} the series would converge. The actual denominator, which is $p(k)(k - 5)!$, is bigger than $p(k)$ for $k > 6$, thus the series converges by comparison with the series with terms k^{-2}. The comparison test is the easy way to go here and would, when you have had a little practice, tell you at a glance that the series converges.

To apply the root test, we would compute $(a_k)^{1/k}$. Using $k! = (2\pi k)^{1/2}(k/e)^k$, approximately, and using the fact that $(bk + c)^{1/k}$ tends to 1 as k tends to infinity for any real numbers $b > 0$ and c, we get that $(a_k)^{1/k}$ tends to zero. Thus, the series converges by the root test.

By the ratio test, we must compute a_{k+1}/a_k. We get

$$\frac{a_{k+1}}{a_k} = \frac{k + 4}{(k + 1)^2}$$

which tends to zero as k tends to infinity. Thus, the series converges by the ratio test.

(1)(b) Remember, it is an immediate consequence of DEFINITION 5.24, that if a series with terms a_k converges then the terms a_k must tend to zero as k tends to infinity. The contrapositive is that if a_k does not tend to zero then the series with terms a_k does not converge. In this example, the terms a_k tend to infinity. Thus, this series is divergent. If you like, this can be thought of as comparison with the divergent series with each term 1.

To apply the root test, we compute easily that the limit of $(a_k)^{1/k}$ as k tends to infinity is $2^{1/2}$. The series diverges by the root test.

To apply the ratio test, we compute a_{k+1}/a_k equals

$$2^{1/2} \frac{k^2 + k + 1}{(k + 1)^2 + (k + 1) + 1}$$

which again tends to $2^{1/2}$ and again implies divergence.

(1)(c) If a_k denotes the k^{th} term of this series, then it is obvious that a_k tends to 1/2 as k tends to infinity. The series diverges. The root test and the ratio test both yield limits of 1 and hence no conclusion can be drawn from them. Check this out by computing the limits in both cases.

(1)(d) An exponential function a^x, $a > 1$, grows much faster than any polynomial of any degree. In particular, $p(x)/a^x$ goes to zero as x tends to infinity, for any fixed polynmial p(x) and any $a > 1$. We can write the term $c_k = k^5 5^{-k}$ as $(k^5 5^{-k/2})5^{-k/2}$. The expression $k^5 5^{-k/2}$ tends to zero as k tends to infinity and thus the series converges by comparison with the geometric series with terms $a_k = 5^{-k/2}$ (take $r = 5^{-1/2}$ in the geometric series). Stop now and reread 5.33 PROBLEMS BASED ON THE COMPARISON TEST. Using that notation, we have c_k/a_k tends to zero and hence the series with terms c_k converges.

The root test is easy for this problem. The numerator of $(c_k)^{1/k}$ is $k^{5/k}$ and the denominator is $5^{k/k} = 5$. The numerator tends to 1 and the denominator, of course, tends to 5. Thus the limit of $(c_k)^{1/k}$ is 1/5 and the series converges.

By the ratio test, we get that $c_{k+1}/c_k = (1/5)(k + 1)^5/k^5$ which again tends to 1/5. It is a theorem that if the limit of $(c_k)^{1/k}$ exists then the limit of c_{k+1}/c_k exists and is the same. The converse, as we noted in 5.37 PROBLEMS BASED ON THE RATIO TEST, is false.

Solutions To (2) Of EXERCISE 5.42

(2)(a) For large values of k, $(k^2 - 150)^{-1}$ "behaves like" k^{-2}, so this series with terms $(k^2 - 150)^{-1}$ converges absolutely by comparison with the series with terms k^{-2}. To be more precise about what we mean by "behaves like," let $c_k = 1/(k^2 - 150)$ and let $a_k = 1/k^2$. The sequence c_k/a_k converges to 1 and is consequently a bounded sequence. The series with terms a_k converges absolutely and thus, by the discussion of 5.33 PROBLEMS BASED ON THE COMPARISON TEST, the series with terms $c_k = 1/(k^2 - 150)$ converges absolutely.

(2)(b) This problem is another example of the idea discussed in connection with the solution to problem (b) of (1). The exponential function 2^k grows much faster than k^2. The terms of this series behave like 2^{-k} and hence this series converges absolutely by comparison with the geometric series $1 + 2^{-1} + 2^{-2} + \ldots$. To apply the discussion of 5.33 PROBLEMS BASED ON THE COMPARISON TEST directly, we let $c_k = (2^k - k^2)^{-1}$ and $a_k = 2^{-k}$. The ratio c_k/a_k tends to 1 as k tends to infinity. The series with terms a_k converges absolutely and hence so does the series with terms c_k.

(2)(c) In the polynomial $k^3 - k^2 - 1$, the term k^3 is dominant for large k. Thus $(k^3 - k^2 - 1)^{1/2}$ behaves like $k^{3/2}$ for large k, and the series of this problem converges absolutely by comparison with the series with terms $k^{-3/2}$. In applying the discussion of 5.33 PROBLEMS BASED ON THE COMPARISON TEST, take $a_k = k^{-3/2}$ and $c_k = 1/(k^3 - k^2 - 1)^{1/2}$. The sequence c_k/a_k converges to 1 and the series with terms a_k converges absolutely.

(2)(d) Multiply the numerator and denominator of the k^{th} term of this series by $[(k + 1)^{1/2} + (k - 1)^{1/2}]$ to get the same series with the k^{th} term now written $2k^{-1}[(k + 1)^{1/2} + (k - 1)^{1/2}]^{-1}$. This term behaves like $4k^{-3/2}$ for large k and hence the series of this problem converges absolutely. We leave it to you to be more precise about "behaves like" for this problem.

Solutions To (3) Of Exercise 5.42

(3)(a) We are going to apply ABEL'S TEST with

$$a_k = 1 + 2^{-2} + 3^{-2} + \ldots + k^{-2}.$$

We showed, using the integral test, that the series

$$1 + 2^{-2} + 3^{-2} + \ldots + k^{-2} + \ldots$$

converged. Hence, the a_k, which are the partial sums of this series, form a nondecreasing sequence with a limit. Taking $b_k = (-1)^k(1/k)$ and applying ABEL'S TEST gives that the series of this problem converges.

(3)(b) This problem looks a lot like part **(a)** except that the sequence of terms $s_k = 1 + (1/2) + \ldots + (1/k)$ diverges, being the partial sums of the harmonic series. This looks bad for ABEL'S TEST. From the integral test, we know that s_k is approximately $\ln(k)$. Since $\ln(k)/k$ goes to zero, there may be hope for DIRICHLET'S TEST. Define $a_k = s_k/k$. The fact that $\ln(k)/k$ goes to zero implies that a_k goes to zero. Here's a computer program to compute the a_k.

```
10 K=1:SK=0
20 SK=SK+1/K
30 PRINT K,SK/K
40 K=K+1
50 GOTO 20
```

k	a_k
1	1
2	.75
3	.6111111
4	.5208334
5	.4566667
6	.4083334
7	.3704082
8	.3397322
9	.3143298
10	.2928968
11	.2745343
12	.2586009
13	.2446257
14	.2322545
15	.2212153
16	.2112956
17	.2023266

18	.1941727
19	.1867231
20	.179887
21	.1735885
22	.1677643
23	.1623605
24	.1573316
25	.1526383
26	.1482469
27	.144128
28	.1402561
29	.1366088
30	.1331662

It looks like a_k is monotonically decreasing to zero. By DIRICHLET'S TEST or by THEOREM 5.22, this means that the series of this problem converges. Actually, we can show by direct computation that the a_k are monotonically decreasing to zero. First, write

$$a_k - a_{k+1} = \frac{1}{k}(1 + (1/2) + \ldots + (1/k)) - \frac{1}{k+1}\left(1 + (1/2) + \ldots + (1/k) + (1/(k+1))\right).$$

Now write this expression as

$$\frac{1}{k(k+1)}\left((k+1)(1 + (1/2) + \ldots + (1/k)) - k(1 + (1/2) + \ldots + (1/k) + (1/(k+1)))\right).$$

Simplifying this expression gives

$$a_k - a_{k+1} = \frac{1}{k(k+1)}\left((1 + (1/2) + \ldots + (1/k)) - \frac{k}{k+1}\right).$$

This shows that $a_k - a_{k+1}$ is greater than zero for all k since there are k terms in the sum $1 + (1/2) + \ldots + (1/k)$ and each term is greater than $1/(k+1)$. That a_k tends to zero follows by comparison with $\ln(k)/k$.

(4) The exponents $(\ln(k))^\beta$ are all positive real numbers but are not integers unless $\beta = 0$, so we should require $r \geqslant 0$ to avoid complex numbers. If $r \geqslant 1$ then the series diverges for all values of β as any positive power of a number greater than or equal to 1 is still greater than or equal to 1. Think of the graph of x^ϵ for $x \geqslant 1$ where $\epsilon > 0$ is small. Thus, we assume $0 \leqslant r < 1$. If $\beta = 0$ then $(\ln(k))^\beta = 1$ for all k and the series diverges unless $r = 0$. If $\beta < 0$ then the sequence of exponents $(\ln(k))^\beta$ tends to 0 and, unless $r = 0$, the terms of the series tend to 1 which implies divergence. Remember, any series diverges if its terms don't converge to zero. So far, nothing but bad news! If this series converges, we must have $0 \leqslant r < 1$ and $\beta > 0$. Of course, if $r = 0$ the series converges, so let's assume $0 < r < 1$ and $\beta < 0$. Write $r = e^{-\alpha}$ where $\alpha > 0$. Then we have

$$r^{(\ln(k))^\beta} = e^{-\alpha(\ln(k))^\beta} = e^{-(\ln(k))\alpha(\ln(k))^{\beta-1}} = k^{-\alpha(\ln(k))^{\beta-1}}.$$

Clearly, if $0 < \beta < 1$ then $\beta - 1$ is negative and this series diverges (its terms become larger than k^{-1}). If $\beta = 1$ then this series converges only if $\alpha > 1$. If $\beta > 1$ then this series converges for all $\alpha > 0$. Why? Because for $\beta > 1$ the expression $\alpha(\ln(k))^{\beta-1}$ tends to infinity as k tends to infinity. As soon as $\alpha(\ln(k))^\beta$ is bigger than 2, say, we can compare the tail of our series with the series k^{-2}, which we know converges.

To summarize, we have that the series

$$\sum_{k=1}^{\infty} r^{(\ln(k))^\beta} \text{ where } r > 0 \text{ and } \beta \text{ is a real number}$$

converges if $\beta = 1$ and $0 < r < e^{-1}$ or if $\beta > 1$ and $0 < r < 1$. In all other cases, it diverges. Remember this fact! Its very useful for comparison tests.

Do you see how this result contains our previous result about the convergence of the series with terms k^{-p} derived just prior to EXERCISE 5.32? Write

$$\sum_{k=1}^{\infty} k^{-p} = \sum_{k=1}^{\infty} (e^{-p})^{\ln(k)}.$$

By the result we have just obtained, with $r = e^{-p}$, this series converges if $e^{-p} < e^{-1}$ and diverges otherwise. In other words, it converges if $p > 1$ and diverges otherwise.

(5)(a) The graph of the function arccot(x) is shown in FIGURE 2.39. The sequence arccot(k) converges monotonically to zero. Thus, the series of this problem converges by DIRICHLET'S TEST with $a_k = \text{arccot}(k)$ and $b_k = (-1)^k$. But, does this series converge absolutely? In the TABLE OF INTEGRALS, we find that the integral of arccot(x) is xarcctn(x) + (1/2)ln(1 + x^2). Thus the series with terms arccot(k) diverges by the integral test. The series with terms $(-1)^k$arcctn(k) is conditionally convergent.

(5)(b) This looks like DIRICHLET'S TEST again with $b_k = \sin(k)$ and $a_k = |k - 99.5|^{-1}$. The DIRICHLET TEST calls for the a_k to be nonincreasing (or nondecreasing) with limit zero. The a_k of this example in fact increase until $k = 99$. After that, the sequence is nonincreasing and converges to zero. But that's good enough. That means, by the DIRICHLET TEST, that the tail series $a_{99}b_{99} + a_{100}b_{100} + \ldots$ converges. This implies that the whole series converges. But does it converge conditionally or absolutely? Read again the solution to EXERCISE 5.15 (1-j) and you will see that, for the same reason presented there, this series does not converge absolutely.

(5)(c) This is an application of ABEL'S TEST with $a_k = (-1)^k/k$ and $b_k = \cos(1/k)$. As k tends to infinity, $\cos(1/k)$ is nondecreasing and tends to 1. The a_k are the terms of the alternating harmonic series, which converges. This series does not converge absolutely, by comparison with the harmonic series.

(5)(d) Again, we have an application of DIRICHLET'S TEST with $a_k = (-9k^2 - 5)/(k^3 + 1)$ and $b_k = \sin(k)$. The sequence a_k is nondecreasing with limit zero and the sequence b_k has bounded partial sums.

This completes the solutions to EXERCISE 5.42. It's time for the variations. Remember to refer to the corresponding problem in EXERCISE 5.42 if you get stuck. Also, don't forget your computer! With simple programs, often involving no more than five or six lines of code, you can gain much valuable information about the infinite series in these problems.

5.44 VARIATIONS ON EXERCISE 5.42

(1) Apply the root test, the ratio test, and the comparison test to each of the following series:

(a) $$\sum_{k=1}^{\infty} \frac{(k-1)(k-2)(k-3)}{k!}$$

(b) $$\sum_{k=1}^{\infty} \frac{2^{k/2}}{k^{200} + k + 1}$$

(c) $$\sum_{k=1}^{\infty} \frac{k^2 + 1}{2k^2 + 3}$$

(d) $\sum_{k=0}^{\infty} \frac{k^{500}}{(1.01)^k}$

(2) Apply the comparison test to each of the following series:

(a) $\sum_{k=0}^{\infty} \frac{1}{k^{1.0001} - 1.50}$

(b) $\sum_{k=1}^{\infty} \frac{1}{2^k - 3k^{2000}}$

(c) $\sum_{k=2}^{\infty} \frac{1}{(k^3 - k^2 - 1)^{1/4}}$

(d) $\sum_{k=1}^{\infty} \frac{(k+1)^{1/4} - (k-1)^{1/4}}{k}$

(3) Discuss the convergence of the following series

(a) $\sum_{k=2}^{\infty} \frac{(-1)^k}{k}\left(\frac{1}{2(\ln(2))^2} + \frac{1}{3(\ln(3))^2} + \ldots + \frac{1}{k(\ln(k))^2}\right)$

(b) $\sum_{k=1}^{\infty} \frac{(-1)^k}{k}\left(\frac{(\ln(1))^2}{1} + \frac{(\ln(2))^2}{2} + \ldots + \frac{(\ln(k))^2}{k}\right)$

(4) The geometric series is the series $1 + r + r^2 + \ldots + r^k + \ldots$ where r is any real number. The terms of this series are of the form $r^{f(k)}$ where $f(k) = k$. Discuss the following generalization of the geometric series in which $f(k) = k$ is replaced by $f(k) = k^\gamma$, γ a real number:

$$\sum_{k=1}^{\infty} r^{k^\gamma} \text{ where } r > 0.$$

(5) Discuss whether or not the following series converge or diverge. If convergent, specify whether the series is conditionally convergent or absolutely convergent and explain why.

(a) $\sum_{k=1}^{\infty} (-1)^k \ln(k)\operatorname{arccot}(k)$

(b) $\sum_{k=0}^{\infty} \frac{\sin(k)}{\ln|k - 99.5|}$

(c) $\sum_{k=1}^{\infty} \frac{(-1)^k}{k}\cos(k)$

(d) $\sum_{k=0}^{\infty} \frac{-9k^2 - 5}{k^3 + 1}\sin(k)\cos(k)$

5.45 VARIATIONS ON EXERCISE 5.42

(1) Apply the root test, the ratio test, and the comparison test to each of the following series:

(a) $\sum_{k=4}^{\infty} \frac{(k+1)(k+2)(k+3)}{k(k-1)(k-2)(k-3)}$

(b) $\sum_{k=1}^{\infty} \frac{k^2}{2^{\ln(k)}}$

(c) $\sum_{k=1}^{\infty} \frac{\ln(k) + (1/k)}{(\ln(k))^3 + 3}$

(d) $\sum_{k=1}^{\infty} \frac{k^{\ln(k)}}{5^k}$

(2) Apply the comparison test to each of the following series:

(a) $\sum_{k=1}^{\infty} \frac{1}{(\ln(k))^2 - 150}$

(b) $\sum_{k=5}^{\infty} \frac{1}{(\ln(\ln(k))^k}$

(c) $\sum_{k=2}^{\infty} \frac{1}{(k^3 - k^2 - 1)^{1/2.99}}$

(d) $\sum_{k=1}^{\infty} \frac{(k+1)^{1/200} - (k-1)^{1/200}}{k}$

(3) Discuss the convergence of the following series. In each case give a theoretical reason for convergence and divergence and write a program to evaluate the series to back up your conclusion. The series expansion for e^x is given in the appendix, if you think that might help.

(a) $\sum_{k=1}^{\infty} (-1)^k(e^{-1/k} - 1 + 1/k)$

(b) $\sum_{k=1}^{\infty} (e^{-1/k} - 1 + 1/k)$

(4) Discuss convergence of the following series in terms of the real numbers α and β:

$$\sum_{k=1}^{\infty} k^{\alpha(\ln(k))^{\beta}}$$

(5) Discuss whether or not the following series converge or diverge. If convergent, specify whether the series is conditionally convergent or absolutely convergent and explain why.

(a) $\sum_{k=0}^{\infty} (-1)^k(\operatorname{arccot}(k))^{\beta}, \ \beta < 1$

(b) $\sum_{k=0}^{\infty} \frac{\sin(k) - \cos(k)}{|k - 99.5|}$

(c) $\sum_{k=1}^{\infty} \cos(k)\sin(1/k)$

(d) $\sum_{k=0}^{\infty} \frac{-9k^2 - 5}{k^3 + 1}(\sin(k)\cos(k/2) + \cos(k)\sin(k/2))$

5.46 VARIATIONS ON EXERCISE 5.42

(1) Apply the root test, the ratio test, *or* the comparison test to each of the following series:

(a) $\sum_{k=1}^{\infty} \frac{k^{\sqrt{k}}}{k!}$

(b) $\sum_{k=1}^{\infty} \frac{(k/4)^k}{k!}$

(c) $\displaystyle\sum_{k=1}^{\infty} \frac{k!}{k(k+1)\ldots(2k-1)}$

(d) $\displaystyle\sum_{k=1}^{\infty} \frac{(k!)^{2k}}{(k^2)!}$

(2) Apply the comparison test to each of the following series:

(a) $\displaystyle\sum_{k=1}^{\infty} \frac{(2k)!}{k^2(k^2-1)\ldots(k^2-k+1)}$

(b) $\displaystyle\sum_{k=5}^{\infty} \left(\frac{1}{\ln(\ln(k))}\right)^{\ln(k)}$

(c) $\displaystyle\sum_{k=5}^{\infty} \left(\frac{1}{\ln(k)}\right)^{\ln(\ln(k))}$

(d) $\displaystyle\sum_{k=2}^{\infty} \frac{k^{\ln(k)}}{(\ln(k))^k}$

(3) Discuss the convergence of the following series in terms of the real numbers α and β:

$$\sum_{k=4}^{\infty} (\ln(k))^{\alpha(\ln(k))^{\beta}}$$

(4) Discuss convergence of the following series in terms of the real number β:

$$\sum_{k=4}^{\infty} r^{(\ln(\ln(k))^{\beta}}, \; r > 0$$

(5) Discuss whether or not the following series converge or diverge. If convergent, specify whether the series is conditionally convergent or absolutely convergent and explain why.

(a) $\displaystyle\sum_{k=0}^{\infty} (-1)^k(\operatorname{arccot}(k))^{\beta}, \; \beta > 1$

(b) $\displaystyle\sum_{k=0}^{\infty} \frac{\sin(k) - \cos(k)}{\ln(|k - 99.5|)}$

(c) $\sum_{k=1}^{\infty} (\sin(k) - \sin(k + (1/k)))$

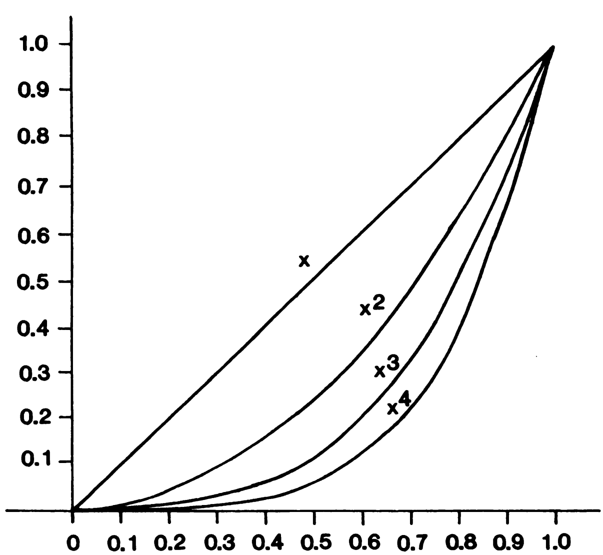
(d) $\sum_{k=0}^{\infty} \ln(1 + (1/k))\sin(2k + \pi)$

Sequences Of Functions

Up to now, we have been concentrating mainly on sequences and series where the terms are real numbers. We began this chapter, however, looking at "power series" approximations to functions. We now return to the idea of series of functions. The idea is simple; we consider a sequence $a_k(x)$, $k = 0, 1, 2, \ldots$, where each $a_k(x)$ is a real valued function of x. For example, $a_k(x)$ could be $\sin(kx)$ or $a_k(x)$ could be x^k. The case $a_k(x) = x^k$ is shown in FIGURE 5.47. Notice that for any particular values of x, say $x = 1/2$, the sequence $a_k(x)$, in this case $a_k(1/2) = (1/2)^k$, becomes a sequence of real numbers. We are experts on such sequences by now, so most of the hard work needed to study sequences of functions is done.

We know that when we study functions, we have to know the domains of the functions. In the sequence of functions $a_k(x)$ of FIGURE 5.47, we have

FIGURE 5.47 The Sequence of Functions x^k, $k = 1, 2, \ldots$



been vague about this. Let's consider two cases. First let's take the domain of $a_k(x)$ to be the interval $[0,1/2] = \{x:0 \leqslant x \leqslant 1/2\}$ for all $k = 1, 2, 3, \ldots$. For each x in the interval [0,1/2], the sequence of numbers $a_k(x)$ converges to zero. We say that the function $a(x) = 0$ is the "limit of the sequence of functions $a_k(x)$ on the interval [0,1/2]."

x^k, $k = 0, 1, 2, \ldots$, Converges Uniformly On [0,1/2]

For the second case, let's take the domains of the $a_k(x)$ to be the interval $[0,1) = \{x:0 \leqslant x < 1\}$. It is still the case that for each x in [0,1), $a_k(x)$ converges to zero. Thus the function $a(x) = 0$ on [0,1) is still the limit of the sequence $a_k(x)$. But, there is something interesting and very important about the difference between these two examples. In the first case, where the common domain is the interval [0,1/2], given any $\epsilon > 0$, if we choose N such that $1/2^N < \epsilon$, then for all $k > N$ and all x in [0,1/2], $|a_k(x)| < \epsilon$. The key phrase here is "and all x in [0,1/2]." A glance at FIGURE 5.47 will explain why this is true. Each $a_k(x)$ has $a_k(1/2)$ as its maximum value over the interval [0,1/2]. Thus, if we make this maximum value small, all other values of $a_k(x)$ on the interval [0,1/2] will be even smaller!

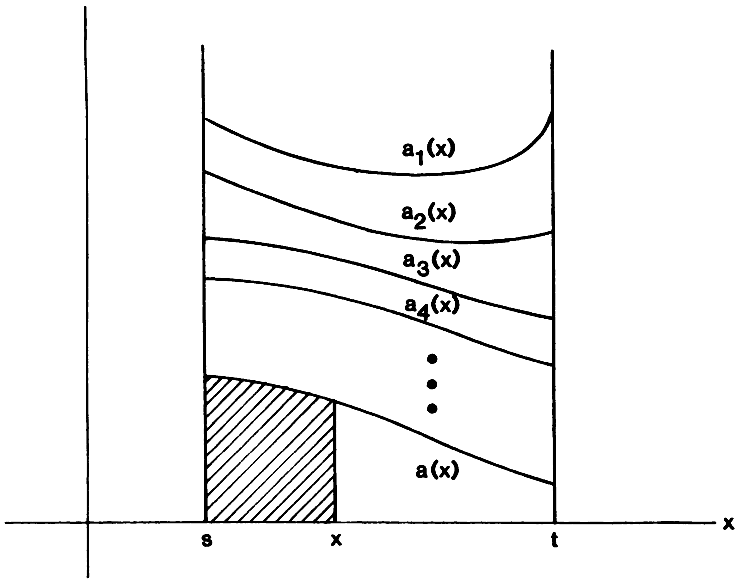
x^k, $k = 0, 1, 2, \ldots$, Does Not Converge Uniformly On [0,1)

But what if we try to play the same game with the common domain of the sequence taken to be the interval [0,1)? It won't work. Even though the sequence of functions $a_k(x)$ converges to the function $a(x) = 0$ at each x in this interval, we can't claim that given any $\epsilon > 0$ there is an N such that, for all $k > N$ and all x in [0,1), $|a_k(x)| < \epsilon$. Do you see why? Look at FIGURE 5.47. Given $\epsilon = 1/2$, for example, and any k, no matter how large, there will always be some x in the interval [0,1), perhaps very close to 1, with $a_k(x) > \epsilon$. This example leads to the following *extremely important* definition.

The Definition Of Uniform Convergence

5.48 DEFINITION Let $a_k(x)$ be a sequence of real valued functions defined on a common domain D. Suppose that for each x in D, the sequence $a_k(x)$ converge to a(x). If, given any $\epsilon > 0$, it is possible to choose N_ϵ such that for all $k > N_\epsilon$ *and for all* x *in* D, $|a_k(x) - a(x)| < \epsilon$, then we say that $a_k(x)$ converges *uniformly* to a(x) on D.

FIGURE 5.49 A Uniformly Convergent Sequence



The Intuitive Idea Of Uniform Convergence

We now want to understand the intuitive idea behind DEFINITION 5.48. Look at FIGURE 5.49. There you see a sequence of functions, $a_k(x)$, $k = 1, 2, 3, 4, \ldots$, which we imagine converging uniformly to a limit function $a(x)$ on the interval $[s,t]$. The graph of the function $a(x)$ is shown by a black line which has a certain thickness (or else you couldn't see it). Let's call the thickness of this line ϵ. According to DEFINITION 5.48, there is an integer N such that for all $k > N$, $|a_k(x) - a(x)| < \epsilon$. Intuitively, this means that for all $k > N$, the lines representing the graphs of the functions $a_k(x)$ completely disappear into the black line that represents the graph of $a(x)$. This is the idea behind the following theorem:

The Limit Of The Integrals Is The Integral Of The Limit

5.50 THEOREM Let $a_k(x)$, $k = 0, 1, 2, \ldots$, be a sequence of real valued continuous functions that converge uniformly on the interval $[s,t]$ to a function $a(x)$. Define functions $A_k(x)$, $k = 0, 1, 2, \ldots$, and $A(x)$ by

$$A_k(x) = \int_s^x a_k(t)dt,\ k = 0, 1, 2, \ldots, \text{ and } A(x) = \int_s^x a(t)dt.$$

Then the sequence of functions $A_k(x)$, k = 0, 1, 2, . . ., converges uniformly to A(x) on [s,t].

The Intuitive Idea Of Theorem 5.50

To understand the meaning of THEOREM 5.50, we refer again to FIGURE 5.49. The shaded area under the graph of the function a is the area under that function and above the interval [s,x]. This area is the integral.

$$\int_s^x a(t)dt$$

which is a function of its upper limit x as x varies in the integral [s,t]. For notational simplicity, let's define, as in THEOREM 5.50,

$$A(x) = \int_s^x a(t)dt \text{ and } A_k(x) = \int_s^x a_k(t)dt.$$

We know that, for all $k > N$, the graphs of the functions $a_k(x)$ disappear into the black line representing the graph of a(x). This means that, for all $k > N$ and all x in [s,t], the difference $|A_k(x) - A(x)|$ is less than the area of the black line used to draw the graph of the function a from x = s to x = t. Theoretically, this black line can be made as thin as we wish to imagine it. This is the intuitive meaning behind uniform convergence of the sequence $A_k(x)$ to A(x) for x in [s,t]. In more advanced courses in analysis, students learn a precise statement and proof of THEOREM 5.50. They study "pathologies" of the functions $a_k(x)$. The functions might be discontinuous, so badly so that they don't have integrals in some sense, etc. We don't have to worry about such strange things in our brief introduction to sequences of functions. Knowing the intuitive idea behind the very useful THEOREM 5.50 is 99 percent of what is required to use it intelligently.

One way to paraphrase THEOREM 5.50 is "For uniformly convergent sequences of functions, the integral of the limit is the limit of the integrals." There is a corresponding result that, loosely stated, says: "For uniformly convergent sequences of functions, the derivative of the limit is the limit of the derivatives." We have to be more careful about this statement. Here is a more precise version which concerns the sequence of derivative functions $a_k'(x)$ of a sequence of functions $a_k(x)$.

The Limit Of The Derivative Is The Derivative Of The Limit

5.51 THEOREM Let $a_k(x)$, k = 0, 1, 2, . . ., be a sequence of real valued functions that converge uniformly on the interval [s,t] to a function a(x). If the sequence of derivative functions $a'_k(x)$, $k = 0, 1, 2, \ldots$, converges uniformly to a function b(x) on [s,t], then b(x) is the derivative function $a'(x)$ for $s < x < t$.

There are technical difficulties with THEOREM 5.51 that must be studied in more advanced courses. The main difference that we should be aware of between THEOREM 5.51 and THEOREM 5.50 is that in THEOREM 5.51, the uniform convergence of the derivative sequence must be verified in each case and does not follow generally from uniform convergence of the original sequence $a_k(x)$. It is not too hard to imagine why this is the case in terms of FIGURE 5.49. Imagine for $k > N$ that all of the functions $a_k(x)$ have disappeared into the black line that represents the graph of a(x), $s < x < t$. As we have noted already, the integral of these "disappearing functions" can't differ from each other by more than the area of the black line. But we could imagine that these functions inside the black line could still be very wiggly, which means that their derivatives might oscillate wildly and differ a lot from each other. This is why the uniform convergence of the derivative functions must be a part of the hypothesis in THEOREM 5.51.

Examples Of Theorems 5.50 And 5.51

Let's test out the ideas of THEOREM 5.50 and THEOREM 5.51 on the sequence $a_k(x) = x^k$ of FIGURE 5.47. In this example, we have uniform convergence on any interval [0,t] where $t < 1$. The integral $A_k(x) = x^{k+1}/(k + 1)$. THEOREM 5.50 says that these functions $A_k(x)$ converge uniformly on [0,t] to the integral A(x) of the function $a(x) = 0$. Of course, $A(x) = 0$ also and it is obviously true that the $A_k(x)$ converge uniformly on [0,t] to the zero function. The derivative functions $a'_k(x) = kx^{k-1}$ can easily be shown to converge uniformly to the function $a'(x) = 0$ on [0,t] if $t < 1$. This is because the maximum of $a'_k(x)$ on the interval [0,t] occurs at t and has value kt^{k-1}. If $0 \leq t < 1$ then this sequence converges to zero. This verifies THEOREM 5.51 for the sequence x^k on [0,t].

To see what can happen if we don't have uniform convergence, consider the sequence $b_k = kx^{k-1}$, $k = 1, 2, \ldots$, on the interval [0,1). The functions kx^{k-1} were called $a'_k(x)$ in the previous paragraph, but we change the name to $b_k(x)$ to emphasize the new domain [0,1). On [0,1), the sequence $b_k(x)$ converges to $b(x) = 0$ at each x, but the convergence is not uniform (why?). The integrals

$$B_k(x) = \int_0^x b_k(t)dt = x^k$$

do not converge uniformly to the zero function on [0,1). In particular, the area under each curve $b_k(x) = kx^{k-1}$, for $0 < x < 1$, is 1. The area under the limit function b(x) is 0. Thus, the limit of the integrals need not be the integral of the limit for non-uniformly convergent sequences.

We now must consider infinite series of functions. Just as with sequences of real numbers, we start with a sequence $a_k(x)$, k = 0, 1, 2, . . ., of functions defined on a common domain D and consider the sequence $s_n(x) = a_0(x) + a_1(x) + \ldots + a_n(x)$, n = 0, 1, 2, . . ., of partial sums with terms from this sequence. This sequence of partial sums is the *infinite series* with terms from the sequence $a_k(x)$, k = 1, 2, ... Just as before, we use without serious harm, the notation

$$\sum_{k=0}^{\infty} a_k(x)$$

to mean both the series and its limit function, depending on the discussion.

If the sequence of partial sums $s_n(x)$ converges uniformly to a limit on the domain D, then we say that the series with terms from the sequence $a_k(x)$, k = 0, 1, 2, . . ., is *uniformly convergent*. The following definition, technically a theorem in a more advanced course, corresponds to DEFINITION 5.24.

Uniform Convergence Of Series

5.52 DEFINITION (UNIFORM CONVERGENCE OF SERIES) The infinite series $a_0(x) + a_1(x) + \ldots$ converges *uniformly* on the domain D if, given any $\epsilon > 0$, there exists an N_ϵ such that for all $q \geq p > N_\epsilon$ and for all x in D, $|a_p(x) + \ldots + a_q(x)| < \epsilon$.

The key phrase in DEFINITION 5.52 is "for all x in D." Through our practice with infinite series of numbers, we have already acquired the technical skills needed to deal with infinite series of functions. Our main concern with series of functions will be how to deal with the important issue of uniform convergence. The next theorem gives a simple but very useful test for uniform convergence.

5.53 THEOREM (WEIERSTRASS'S M TEST) Let $a_0(x) + a_1(x) + \ldots$ be an infinite series of functions defined on a domain D. Let M_k be a sequence of numbers such that $|a_k(x)| \leq M_k$ for all x in D. If the series $M_0 + M_1 + \ldots$ converges then the series $a_0(x) + a_1(x) + \ldots$ converges uniformly on D.

Proof: Given any $\epsilon > 0$, we can choose N_ϵ such that for all $q \geqslant p > N_\epsilon$, $|M_p + \ldots + M_q| = M_q + \ldots + M_p < \epsilon$. But $|a_q(x) + \ldots + a_p(x)| \leqslant |a_p(x)| + \ldots + |a_q(x)| \leqslant M_q + \ldots + M_p$ for all x in D. Thus the sequence $a_0(x) + a_1(x) + \ldots$ converges uniformly on D by DEFINITION 5.52. This completes the proof.

You can see from the proof of THEOREM 5.53 that, in fact, the series of absolute values, $|a_0(x)| + |a_1(x)| + \ldots$, converges uniformly on D. The series with terms $a_k(x)$, $k = 0, 1, 2, \ldots$, is thus "absolutely uniformly convergent." Notice also that if the sequence M_k, $k = 0, 1, 2, \ldots$, is replaced by a sequence of functions $M_k(x)$, $k = 0, 1, 2, \ldots$, which converge uniformly on D and satisfy $|a_k(x)| \leqslant M_k(x)$ for all x in D, the proof is still valid.

As a simple example of the M-test, consider the series

$$\sum_{k=1}^{\infty} \frac{x^k}{k^\alpha}, \quad \alpha > 1.$$

Let $M_k = \dfrac{1}{k^\alpha}$. Take the domain D of the functions x^k/k^α to be $[-1,+1]$. On the interval $[-1,+1]$, $|x^k/k^\alpha| \leqslant M_k$. Thus the convergence of the series with terms M_k, $k = 1, 2, \ldots$, implies the uniform convergence of the series of functions with terms x^k/k^α on the interval $[-1,+1]$.

The analogs THEOREMS 5.50 and 5.51, which concern integrals and derivatives of sequences of functions, follow immediately. The series with terms from the sequence of functions $a_k(x)$, $k = 0, 1, 2, \ldots$, is, by definition, the sequence of partial sums $s_n(x)$, $n = 0, 1, 2, \ldots$, of this sequence. Suppose the domain D of these functions is the interval [s,t]. Suppose that the sequence $s_n(x)$ converges uniformly on [s,t] to s(x). By definition, this means that the series with terms $a_k(x)$, $k = 1, 2, \ldots$, converges uniformly on [s,t]. Define functions $S_n(x)$ and S(x) on [s,t] by

$$S_n(x) = \int_s^x s_n(t)dt \quad \text{and} \quad S(x) = \int_s^x s(t)dt.$$

By THEOREM 5.50, the sequence $S_n(x)$ converges uniformly to S(x) on the interval [s,t].

The previous paragraph contains all of the ideas needed to apply THEOREM 5.50 to series. It is necessary, however, to fool around with the notation a bit to make sure you can recognize the various possible ways of saying the same thing. For example, we can write

$$S(x) = \int_s^x \left(\sum_{k=0}^{\infty} a_k(t) \right) dt \text{ and}$$

$$S_n(x) = \int_s^x s_n(t)dt = \int_s^x \left(\sum_{k=0}^{n} a_k(t) \right) dt = \sum_{k=0}^{n} \int_s^x a_k(t)dt.$$

The last of the above expressions for $S_n(x)$ is a common way of thinking about $S_n(x)$. Starting with the sequence of functions $a_k(x)$, $k = 0, 1, 2, \ldots$, we form the new sequence of functions

$$\int_s^x a_k(t)dt \text{ , } k = 0, 1, 2, \ldots .$$

This sequence is the sequence of integrals of the terms of the series $a_0(x) + a_1(x) + \ldots$. Doing this is called "term by term integration of the series." $S_n(x)$ is the sequence of partial sums of these integrals. Here is a common way to state THEOREM 5.50 applied to series when using this type of notation.

Term-By-Term Integration

5.54 THEOREM Suppose that the series of functions of x

$$\sum_{k=0}^{\infty} a_k(x)$$

converges uniformly on the interval [s,t]. Then the series of functions of x

$$\sum_{k=0}^{\infty} \left(\int_s^x a_k(t)dt \right)$$

converges uniformly on [s,t], and

$$\sum_{k=0}^{\infty} \left(\int_s^x a_k(t)dt \right) = \int_s^x \left(\sum_{k=0}^{\infty} a_k(t) \right) dt.$$

The analog of THEOREM 5.51, concerning term-by-term differentiation is as follows:

Term-By-Term Differentiation

5.55 THEOREM Suppose that both of the series

$$\sum_{k=0}^{\infty} a_k(x) \qquad \text{and} \qquad \sum_{k=0}^{\infty} a_k'(x)$$

converge uniformly on [s,t]. The for x in [s,t],

$$\frac{d}{dx}\sum_{k=0}^{\infty} a_k(x) = \sum_{k=0}^{\infty} a_k'(x).$$

We began this chapter by working with infinite series of functions called power series. We didn't really know what we were doing, but it was fun. In particular, we did term-by-term integration on power series. For example, at one point we had the power series

$$\frac{\sin(x)}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \ldots$$

We integrated this power series term by term. We knew it was o.k. to do this because we checked our answer on the computer by other means (a Riemann sum). The limits of integration that we used were from 0 to π. According to THEOREM 5.54, integration term by term is all right if this series converges uniformly on the interval $[0,\pi]$. Not only must the series converge uniformly on $[0,\pi]$, but it must converge to sin(x)/x for us to get the correct answer.

When a series does or does not converge to a given function is another matter that we will take up below. To show that the series for sin(x)|x converges uniformly on $[0,\pi]$, we can use the M-test (THEOREM 5.53) with $M_k = \pi^k/(k+1)!$ Each term in the series satisfies $|x^k/(k+1)!| \leq M_k$ for $0 < x < \pi$. The series with terms M_k, k = 0, 2, 4, . . ., is easily seen to converge (root test, ratio test, or comparison test) and hence the given series for sin(x)/x converges uniformly on $[0,\pi]$. The same argument shows that it converges uniformly on any interval $[-c,+c]$ for any real number $c > 0$.

Power Series: $\sum c_k(x - a)^k$

In general, a power series is a series of functions $a_k(x)$, k = 0, 1, 2, . . ., where $a_k(x) = c_k(x - a)^k$ and the c_k, called the "coefficients of power series," are real numbers and a is a real number.

5.56 DEFINITION (Power Series) A series of functions of the form

$$\sum_{k=0}^{\infty} c_k(x - a)^k$$

where a and c_k, $k = 0, 1, 2, \ldots$ are real numbers, is called a *power series*.

Suppose we try to apply the root test to the power series with terms $c_k(x - a)^k$. We compute the limit

$$\operatorname*{limit}_{k\to\infty} |c_k(x - a)^k|^{1/k} = \operatorname*{limit}_{k\to\infty} |c_k|^{1/k} |x - a| = L|x - a|.$$

The number L is the limit as k goes to infinity of $|c_k|^{1/k}$. The series converges if $|x - a| < L^{-1}$ and diverges if $|x - a| > L^{-1}$. The number $R = L^{-1}$ is called the "radius of convergence" of the power series. Except for some technical details (the limit of $|c_k|^{1/k}$ might not exist), we have proved the following theorem.

5.57 THEOREM Let

$$\sum_{k=0}^{\infty} c_k(x - a)^k$$

be a power series and let L be the limit of $|c_k|^{1/k}$. The number $R = L^{-1}$ is called the radius of convergence of the power series. The power series converges absolutely for all numbers x such that $|x - a| < R$ and diverges for $|x - a| > R$. For $|x - a| = R$ the power series may either converge or diverge, depending on the particular case.

For example, if $c_k = 2^{-k}$ then $L = 1/2$. Thus the radius of convergence of the power series

$$\sum_{k=0}^{\infty} 2^{-k}(x - a)^k$$

is 2. If $c_k = 1/k!$, then $L = 0$. In this case, we say that the radius of convergence is infinite. The power series

$$\sum_{k=0}^{\infty} (x - a)^k/k!$$

converges for all x.

The Proof Of Taylor's Theorem

The next result concerns power series approximations to functions. We shall prove a theorem called "TAYLOR'S THEOREM." This theorem is very useful and has an unforgettably simple proof based on integration by parts. Recall that, just prior to EXERCISE 4.26, we introduced the following tabular notation or "box notation" for describing integration by parts:

$f(x)$	$g(x)$
$f'(x)$	$g'(x)$

$$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx$$

In words, the integral of the product of the entries on either diagonal is the product of the entries in the top row minus the integral of the product of the entries on the other diagonal.

Suppose we have a function f(x) that is "smooth" in the sense that it has lots of derivatives. Most functions we use in calculus are like that. They can be differentiated over and over again. The functions sin(x), tan(x), cos(x), ln(x), e^x, cosh(x), arcsin(x) $(1 + x)^{1/2}$, etc., have this property for many values of x.

Let b be any real number, and consider the following sequence of boxes with their corresponding integration by parts formulas:

$f^{(1)}(x)$	$b - x$
$f^{(2)}(x)$	-1

$$\int f^{(1)}(x)d(b - x) = f^{(1)}(x)(b - x) + \int f^{(2)}(x)(b - x)d(b - x)$$

$f^{(2)}(x)$	$\dfrac{(b - x)^2}{2!}$
$f^{(3)}(x)$	$-(b - x)$

$$\int f^{(2)}(x)(b-x)d(b-x) = f^{(2)}(x)\frac{(b-x)^2}{2!} + \int f^{(3)}(x)\frac{(b-x)^3}{3!}d(b-x)$$

$f^{(3)}(x)$	$\dfrac{(b-x)^3}{3!}$
$f^{(4)}(x)$	$-\dfrac{(b-x)^2}{2!}$

$$\int f^{(3)}(x)\,\frac{(b-x)^2}{2!}\,d(b-x) = f^{(3)}(x)\,\frac{(b-x)^3}{3!} + \int f^{(4)}(x)\,\frac{(b-x)^3}{3!}\,d(b-x)$$

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.
.

(general case)

$f^{(n)}(x)$	$\dfrac{(b-x)^n}{n!}$
$f^{(n+1)}(x)$	$-\dfrac{(b-x)^{n-1}}{(n-1)!}$

$$\int f^{(n)}(x)\,\frac{(b-x)^{n-1}}{(n-1)!}\,d(b-x) = f^{(n)}(x)\,\frac{(b-x)^n}{n!} + \int f^{(n+1)}(x)\,\frac{(b-x)^n}{n!}\,d(b-x).$$

In the above sequence of boxes, focus your attention on the boxes, not the formulas. It's easier to remember that way, and the formulas follow automatically. In these formulas, the minus sign that usually appears in the last integral of the integration by parts formula is incorporated into the $d(b - x)$. Starting with the first formula, replace the last integral in each formula with the expression given for it in the next formula to get

$$\int f^{(1)}(x)d(b-x) =$$

$$f^{(1)}(x)(b-x) + f^{(2)}(x)\,\frac{(b-x)^2}{2!} + \ldots + f^{(n)}(x)\,\frac{(b-x)^n}{n!}$$

$$+ \int f^{(n+1)}(x)\,\frac{(b-x)^n}{n!}\,d(b-x).$$

This expression is an exact statement about indefinite integrals. It looks useless at first glance. However, now let's compute the definite integral

$$\int_b^a f^{(1)}(x)d(b - x) = f(b) - f(a).$$

The above formula now becomes

$$f(b) - f(a) =$$

$$f^{(1)}(a)(b - a) + f^{(2)}(a)\frac{(b - a)^2}{2!} + \ldots + f^{(n)}(a)\frac{(b - a)^n}{n!}$$

$$+ \int_b^a f^{(n+1)}(x)\frac{(b - x)^n}{n!}d(b - x).$$

By writing d(b − x) = dx and reversing the order of the limits of integration, this same formula can be written

$$f(b) = f(a) + f^{(1)}(a)(b - a) + f^{(2)}(a)\frac{(b - a)^2}{2!} + \ldots + f^{(n)}(a)\frac{(b - a)^n}{n!}$$

$$+ \int_a^b f^{(n+1)}(x)\frac{(b - x)^n}{n!}dx.$$

The above formula is not an approximation. It is an identity valid for any function f with n + 1 continuous derivatives in some interval that contains a and b. In this identity, you can imagine the number a as fixed and b as a variable. When thinking of the identity in this way, it is common to replace the symbol b by x and write

$$f(x) = f(a) + f^{(1)}(a)(x - a) + f^{(2)}(a)\frac{(x - a)^2}{2!} + \ldots + f^{(n)}(a)\frac{(x - a)^n}{n!}$$

$$+ \int_a^x f^{(n+1)}(t)\frac{(x - t)^n}{n!}dt.$$

We have changed the variable from x to t in the definite integral, to avoid confusion with x, the new name for b. We introduce the following standard terminology.

Taylor Polynomial

5.58 DEFINITION Let f(x) be a function with n + 1 derivatives at a point a. The polynomial

$$T_n(x) = f(a) + f^{(1)}(a)(x-a) + f^{(2)}(a)\frac{(x-a)^2}{2!} + \ldots + f^{(n)}(a)\frac{(x-a)^n}{n!}$$

is called the "Taylor polynomial of degree n of f at a."

Putting these ideas together, we have the following important theorem.

5.59 THEOREM (Taylor's Theorem) Let f be a function with continuous derivative function $f^{(n+1)}$ in the interval $I = (\alpha,\beta) = \{x:\alpha < x < \beta\}$. For any a and x in I

$$f(x) = T_n(x) + \int_a^x f^{(n+1)}(t)\frac{(x-t)^n}{n!}\,dt \text{ where}$$

$$T_n(x) = f(a) + f^{(1)}(a)(x-a) + f^{(2)}(a)\frac{(x-a)^2}{2!} + \ldots + f^{(n)}(a)\frac{(x-a)^n}{n!}$$

is the TAYLOR POLYNOMIAL of degree n of f at a.

The expression

$$R_{n+1}(x,a) = \int_a^x f^{(n+1)}(t)\frac{(x-t)^n}{n!}\,dt$$

is called the *remainder in the Taylor polynomial approximation to* f(x). This integral represents the amount f(x) differs from its Taylor polynomial of degree n at a. If for fixed x and a, the sequence $R_n(x,a)$ tends to zero as n tends to infinity, then we write

$$f(x) = \sum_{k=0}^{\infty} f^{(k)}(a)\frac{(x-a)^k}{k!}.$$

The power series

$$\sum_{k=0}^{\infty} f^{(k)}(a)\frac{(x-a)^k}{k!}$$

is called the "Taylor series of f at a."

There are two different questions that can be asked about the Taylor series of f at a. The first question is "For what values of x does the Taylor series of f at a converge?" The second question is "For what values of x does the Taylor series of f at a converge to f(x)?" For example, in the math tables in the appendix we find the statement

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots \qquad [x^2 < \infty]$$

This statement conveys the information that the Taylor series of sin(x) at $a = 0$ is

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$$

and, by the statement $[x^2 < \infty]$, that this series converges to sin(x) for all x. The function $z(x) = e^{-1/x^2}$, defined to be zero at $x = 0$, has derivatives of all orders, $z^{(n)}(x)$, for all values of x, including $x = 0$ (you should show this to be true). At 0, $z^{(n)}(0) = 0$, for all $n = 0, 1, 2, \ldots$. Thus the Taylor series of this function at $a = 0$ is the identically zero series. The Taylor series of this function at $a = 0$ obviously converges for all x but is equal to z(x) only at $x = 0$. Using z(x), you can construct other curious examples of Taylor series. The function $g(x) = \sin(x) + z(x)$ has the same Taylor series as sin(x) at $a = 0$. This Taylor series converges everywhere but converges to g(x) only at $x = 0$. The function $h(x) = z(x)\sin(x) + \sin(x)$ has the same Taylor series as sin(x) at $a = 0$ but converges to h(x) only at $x = \pi k$, $k = 0, \pm 1, \pm 2, \ldots$.

Estimating The Remainder In Taylor's Theorem

In regard to the second question, "For what values of x does the Taylor series of f at a converge to f(x)?", it is helpful to have some techniques for showing convergence of the series of remainders $R_n(x,a)$. One useful technique supposes that we know an upper bound for $f^{(n+1)}(t)$ on the interval [a,x].

5.60 REMAINDER ESTIMATE Suppose that $|f^{(n+1)}(t)| \leq B_{n+1}$ for all t in the interval [a,x]. Then

$$|R_{n+1}(x,a)| \leq B_{n+1} \frac{|x - a|^{n+1}}{(n + 1)!} .$$

Proof: We see from the form of $R_{n+1}(x,a)$ given above as a definite integral that $|R_{n+1}(x,a)|$ is less than or equal to the upper bound B_{n+1} times the area between the curve $|x - t|^n/n!$ and the interval $[a,x]$ on the x-axis. This area is exactly $|x - a|^{n+1}/(n + 1)!$ which proves the assertion

There is another common way of expressing the remainder, $R_{n+1}(x,a)$, called LAGRANGE'S FORM OF THE REMAINDER.

5.61 LAGRANGE'S FORM OF THE REMAINDER There exists some number c between a and x such that

$$R_{n+1}(x,a) = f^{(n+1)}(c)\,\frac{(x - a)^{n+1}}{(n + 1)!}\,.$$

The good news about Lagrange's form for the remainder is that it is not an approximation but an equality. The bad news is that we hardly ever are able to find the mysterious number c. By considering all possibilities for c, we can get back to estimating the remainder, as in REMAINDER ESTIMATES 5.60.

As an application of REMAINDER ESTIMATES 5.60, consider the Taylor series for sin(x) at 0:

$$\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k + 1)!}\,.$$

By the ratio test, or the root test together with Stirling's formula, we see easily that this series converges absolutely for all x. In fact, from Weierstrass's M test, it converges uniformly on every interval $[\alpha,\beta]$, where $\alpha < \beta$ are real numbers. The higher order derivatives of sin(x) are all either $+\sin(x)$, $-\sin(x)$, $+\cos(x)$, or $-\sin(x)$. Consequently, the bound $B_{n+1} = 1$ works for all n. Thus, the sequence of remainders, $|R_{n+1}(x,0)|$, is less than or equal to $|x|^{n+1}/(n + 1)!$, which converges to zero for all x. This shows that the Taylor series for sin(x) at 0 converges to sin(x) for all x.

There are three basic operations on general power series that we now must take a look at.

5.62 BASIC OPERATIONS ON A POWER SERIES

(1) Starting with $\sum_{k=0}^{\infty} c_k(x - a)^k$ form $\sum_{k=1}^{\infty} kc_k(x - a)^{k-1}$ by differentiating term by term.

(2) Starting with $\sum_{k=0}^{\infty} c_k(x - a)^k$ form $\sum_{k=0}^{\infty} \frac{c_k}{(k + 1)} (x - a)^{k+1}$ by integrating term by term.

(3) Starting with $\sum_{k=1}^{\infty} c_k(x - a)^k$ form $\sum_{k=1}^{\infty} c_k(x - a)^{k-1}$ by dividing each term by $x - a$.

5.63 THEOREM Each of the power series of 5.62 BASIC OPERATIONS ON A POWER SERIES has the same radius of convergence $R = L^{-1}$ where

$$L = \lim_{k\to\infty} |c_k|^{1/k}.$$

Proof: In case (1) we compute $\lim_{k\to} |kc_k(x - a)^{k-1}|^{1/k} =$

$$\left(\lim_{k\to\infty} (k^{1/k})\right)\left(\lim_{k\to\infty} |c_k|^{1/k}\right)\left(\lim_{k\to\infty} |x - a|^{(k-1)/k}\right) = (1)(L)(|x - a|).$$

Thus, in case (1), the second series converges if $|x - a| < L^{-1} = R$ and diverges if $|x - a| > R$.

In case (2), using the fact that $\lim_{k\to} (k + 1)^{-1/k} = 1$ and

$$\lim_{k\to\infty} |x - a|^{(k+1)/k} = |x - a|,$$

we again get R as the radius of convergence of the second series.

In case (3), we need only that

$$\lim_{k\to\infty} |x - a|^{(k-1)/k} = |x - a|$$

to get R as the radius of convergence. This completes the proof of THEOREM 5.63.

To complete our understanding of BASIC OPERATIONS ON A POWER SERIES we need the next easy result.

Uniform And Absolute Convergence Of Taylor's Series

5.64 THEOREM If R is the radius of convergence of the power series

$$\sum_{k=0}^{\infty} c_k(x - a)^k$$

and $0 < r < R$, then this series converges uniformly on the interval $[a - r, a + r]$.

Proof: Apply the Weierstrass M test with $M_k = |c_k| r^k$.

It obviously follows from THEOREM 5.64 that a power series converges uniformly on any interval $[\alpha,\beta]$, where α and β are contained in the interval of convergence of the power series.

It's time to worry about notation again. If you see a statement like this

$$\text{``Consider the power series } c(x) = \sum_{k=0}^{\infty} c_k(x - a)^k \ldots\text{''}$$

then the symbol c(x) stands for the power series itself. That is, c(x) stands for the sequence of partial sums of the sequence $c_k(x - a)^k$, $k = 0, 1, 2, \ldots$. If, on the other hand, you see a statement like this

$$\text{``Consider the function } c(x) = \sum_{k=0}^{\infty} c_k(x - a)^k \text{ where } |x - a| < R \ldots\text{''}$$

the c(x) stands for the function defined by the limit of the power series for values of x such that $|x - a|$ is less than the radius of convergence R. There is usually little chance of confusing these two different meanings for c(x).

We now summarize the above discussion with an important theorem for working with power series.

Differentiation And Integration Of Power Series

5.65 THEOREM Let R be the radius of convergence of the power series with terms $c_k(x - a)^k$, $k = 0, 1, 2, \ldots$, and define the function

$$c(x) = \sum_{k=0}^{\infty} c_k(x - a)^k, \quad |x - a| < R.$$

Then

(1) $$c'(x) = \sum_{k=1}^{\infty} kc_k(x - a)^{k-1} \text{ for } |x - a| < R$$

(2) $$\int_a^x c(x)\,dx = \sum_{k=0}^{\infty} \frac{c_k}{k + 1}(x - a)^{k+1} \text{ for } |x - a| < R$$

and, if $c(a) = 0$,

(3) $$\frac{c(x)}{x - a} = \sum_{k=1}^{\infty} c_k(x - a)^{k-1} \text{ for } |x - a| < R.$$

Proof: Choose $r < R$ such that $a - r < x < a + r$. All series of the theorem converge uniformly on the interval $[a - r, a + r]$. Then (1) follows from THEOREM 5.51 and (2) follows from THEOREM 5.50. The uniform convergence of the series in (2) is a consequence of THEOREM 5.50, although we have verified uniform convergence directly by THEOREMS 5.63 and 5.64. In the case of (3), $c(a) = 0$ means that $c_0 = 0$. Thus, (3) follows by multiplying both sides by $(x - a)$ and using the basic arithmetic rule (1) for series stated just prior to EXERCISE 5.42.

In the beginning of this chapter, we started with the series for sin(x).

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$$

We then divide by x to obtain

$$\frac{\sin(x)}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \ldots$$

Next we integrate term by term to obtain

$$\int \frac{\sin(x)}{x}\,dx = 1 - \frac{x^3}{3\ 3!} + \frac{x^5}{5\ 5!} - \ldots$$

All of the above operations are now justified by THEOREM 5.65, and all series have the same radius of convergence ($R = \infty$) as the original series.

The operation of differentiation as in THEOREM 5.65(1) can be applied repeatedly, always obtaining a series with the same radius of convergence as the original. When you compute $c^{(n)}(x)$ and evaluate at $x = a$, you obtain $c^{(n)}(a) = n!c_n$. Thus, if a function is defined by a power series in its radius of convergence, such as c(x) was in THEOREM 5.65, that power series is the Taylor series of that function.

Addition And Multiplication Of Power Series

The same argument used to prove THEOREM 5.65(3) shows that if you take any polynomial $p(z) = a_0 + a_1z + a_2z^2 + \ldots + a_nz^n$ and form the product

$$p(x - a)c(x) = p(x - a) \sum_{k=0}^{\infty} c_k(x - a)^k$$

and collect terms on the right according to powers of (x − a), then you obtain the power series for p(x − a)c(x) and that power series converges for $|x - a| < R$.

More generally, you can take any two power series

$$c(x) = \sum_{k=0}^{\infty} c_k(x - a)^k \text{ and } d(x) = \sum_{k=0}^{\infty} d_k(x - a)^k$$

and add them term by term or take their product, expressing the answer as a sum of powers of (x − a). These operations can be done purely "formally" without regard to intervals of convergence. If, however, r is the minimum of the two radii of convergence for these two power series, then c(x) and d(x) are functions defined for $|x - a| < r$. In that case, the sum of the two series converges to the function c(x) + d(x) for $|x - a| < r$ and the product converges to c(x)d(x) for $|x - a| < r$. In either case the actual radius of convergence of the sum series or product series might be larger than r. For example, if the series for c(x) had radius of convergence 1 and you subtract that series from itself, you get the zero series, which has infinite radius of convergence.

As an example of taking the product of two series, consider

$$\sin(x) = \sin(a) + \cos(a)(x - a) - \sin(a)(x - a)^2/2! + \ldots$$

and

$$e^x = e^a + e^a(x - a) + e^a(x - a)^2/2! + \ldots$$

The product

$$e^x\sin(x) = e^a\sin(a) + (e^a\sin(a) + e^a\cos(a))(x - a) + e^a\cos(a)(x - a)^2 + \ldots.$$

Instead of doing such a calculation, you can always compute the Taylor series directly. For $f(x) = e^x\sin(x)$, we find that $f^{(1)}(x) = e^x\sin(x) + e^x\cos(x)$ and $f^{(2)}(x) = 2e^x\cos(x)$. Thus $f^{(1)}(a) = e^a\sin(a)$ and $f^{(2)}(a) = 2e^a\cos(a)$ and we obtain the same series directly from the definition of the Taylor series.

Composing Power Series

Finally, there are the operations of division of power series and composition of power series. Both operations usually involve tedious operations. Modern algebraic symbol manipulation software is a great help with regard to these and all operations on power series. To simplify matters, we shall stick to power series in x. Taylor series expansions of functions about a = 0 are called MACLAURIN SERIES. Let's compute the Maclaurin series of the composition $e^{\sin(x)}$. We have

$$e^x = 1 + x + x^2/2! + x^3/3! + \ldots$$

and

$$\sin(x) = x - x^3/3! + x^5/5! - \ldots.$$

Both series converge for all x. The composition $e^x\sin(x)$ is thus

$$1 + (x - x^3/3! + x^5/5! - \ldots) + (x - x^3/3! + x^5/5! - \ldots)^2/2!$$
$$+ (x - x^3/3! + x^5/5! - \ldots)^3/3! + \ldots$$

Taking powers and collecting terms, we get

$$e^{\sin(x)} = 1 + x + x^2/2! - 3x^4/4! - \ldots$$

What happens with radii of convergence under composition of power series can be quite tricky. A little common sense will avoid the worst blunders. If you are going to compose c(x), which converges for $|x| < R_1$ and d(x) which converges for $|x| < R_2$ and want the power series for c(d(x)) to converge at $x = t$, then you better have both $|t| < R_2$ and $|d(t)| < R_1$. In our example, both power series had infinite radius of convergence. In combinatorial mathematics, composition of power series plays an important role even when the series don't converge. One very common type of composition of series is to compose a series, such as that of sin(x), with a power of x, such as x^2, to obtain a series such as $\sin(x^2) = x^2 - x^6/3! + \ldots$. Here, $c(x) = \sin(x)$ and $d(x) = x^2$.

Dividing Power Series

Dividing one series c(x) by d(x) to obtain the series for c(x)/d(x) is directly analogous to polynomial division and every bit as tedious. Of course, you must take care not to divide by zero. Here we divide the series for $\sin(x^2)$ by cos(x) to obtain a few terms of the series for $\sin(x^2)/\cos(x)$.

$$
\begin{array}{r|l}
 & x^2 + \frac{x^4}{2} + \frac{x^6}{24} + \ldots \\
\hline
1 - \frac{x^2}{2} + \frac{x^4}{24} - \ldots & x^2 \qquad - \frac{x^6}{6} \qquad + \ldots \\
 & x^2 - \frac{x^4}{2} + \frac{x^6}{24} \\
 & \underline{\qquad\qquad\qquad}
\end{array}
$$

$$\frac{x^4}{2} - \frac{5x^6}{24} + \ldots$$

$$\frac{x^4}{2} - \frac{x^6}{4} + \ldots$$

$$\frac{x^6}{24} + \ldots$$

$$\frac{x^6}{24} + \ldots$$

$$0 + \ldots$$

Thus, computed to three terms, we have

$$\frac{\sin(x^2)}{\cos(x)} = x^2 + \frac{x^4}{2} + \frac{x^6}{24} + \ldots . \qquad -\frac{\pi}{2} < x < \frac{\pi}{2} .$$

Limits Of Functions

In DEFINITION 3.11 we defined the notion of a limit of a function f(x) at $x = t$ in terms of limits from the left and limits from the right at t. Using sequences, we can be more precise about these definitions. Let x_k, $k = 0, 1, 2, \ldots$, be an infinite sequence. If the limit as k goes to infinity of x_k is t, we use the shorthand notation $x_k \rightarrow t$. If also $x_x < t$ for all k then we write $x_k \rightarrow t-$ and if $x_k > t$ for all k, we write $x_k \rightarrow t+$. We have the following definition:

5.66 DEFINITION We say that the "limit from the left of f(x) at t is A" and write

$$\underset{x \rightarrow t-}{\text{limit}} f(x) = A$$

if for every sequence $x_k \rightarrow t-$ the sequence $f(x_k) \rightarrow A$. Similarly, we define the "limit from the right of f(x) at t is A" and write

$$\underset{x \rightarrow t+}{\text{limit}} f(x) = A.$$

Combining these ideas, if the limit from the left *and* the right of f(x) at t is A, then we say that "the limit of f(x) at t is A" and write

$$\underset{x \rightarrow t}{\text{limit}} f(x) = A.$$

The Epsilon—Delta Definition Of A Limit

The definition of "the limit of f(x) at t is A" can be stated directly as "for every sequence $x_k \rightarrow t$, $f(x_k) \rightarrow t$." An alternative definition of "the limit of f(x) at t is A" is to say that "For every $\epsilon > 0$ there exists a $\delta > 0$ such that for all x such that $|x - t| < \delta$, $|f(x) - A| < \epsilon$." This definition is called "the epsilon-delta definition of the limit." It is equivalent to the definition of "the limit of f(x) at t is A" given in DEFINITION 5.66. You should be getting to the point now where you can prove that these two definitions are equivalent. Give it a try!

Continuous Functions

Corresponding to DEFINITION 3.12, we have

5.67 DEFINITION We say that f(x) is *left continuous at t* if

$$\underset{x \to t-}{\text{limit}}\ f(x) = f(t).$$

Similarly, we define "f(x) is *right continuous at t*." If f(x) is both left continuous and right continuous at t then we say that f(x) is *continuous at t*.

A direct statement that f(x) is continuous at t is "for every sequence $x_k \rightarrow t$, $f(x_k) \rightarrow f(t)$." Intuitively, continuous functions at t have nice smooth graphs at the point (t,f(t)). The function $f(x) = +1$ if $x \geqslant 0$ and $f(x) = -1$ if $x < 0$ is discontinuous at $x = 0$. It is right continuous at $x = 0$ but not left continuous. At $x = 0$, the graph of this function, which you should sketch, has a "jump discontinuity" at $x = 0$. Imagine a sequence of functions $f_n(x)$ converging uniformly to this function f(x) on $[-1, +1]$. As in FIGURE 5.49, there is some N such that for all $n > N$, the graphs of the functions $f_n(x)$ have disappeared into the line representing the graph of f(x). Thus, for $n > N$, the functions $f_n(x)$ must also be discontinuous at $x = 0$. The idea is that if a sequence of functions $f_n(x)$ converges uniformly to a function f(x) that is discontinuous at t, then all but possibly a finite number of the $f_n(x)$ must also be discontinuous at t. The contrapositive to this statement is as follows:

Uniform Convergence And Continuity

5.68 THEOREM Let $f_n(x)$ be a sequence of functions that converge uniformly to f(x) on some interval $[a - r, a + r]$. If all but possibly a finite number of the functions $f_n(x)$ are continuous at t, $a - r < t < a + r$, then f(x) is continous at t.

Proof: We must show that for any sequence $x_k \rightarrow t$, $f(x_k) \rightarrow f(t)$. The values of x_k may be assumed to be in the interval $(a - r, a + r)$. This is equivalent to showing that for any $\epsilon > 0$, there is some K such that for all $k > K$, $|f(t) - f(x_k)| < \epsilon$. We write

$$|f(t) - f(x_k)| = |f(t) - f_n(t) + f_n(t) - f_n(x_k) + f_n(x_k) - f(x_k)|$$
$$\leqslant |f(t) - f_n(t)| + |f_n(t) - f_n(x_k)| + |f_n(x_k) - f(x_k)|.$$

By uniform convergence, we can, by picking n large enough, make sure that $|f_n(x) - f(x)| < \epsilon/3$ for all x in the interval, in particular for $x = t$ and $x = x_k$, $k = 0, 1, \ldots$. Fix such an n and choose it to be such that $f_n(x)$ is continuous at t (only finitely many f_n are not continuous at t). Now, choose K such that for all $k > K$, $|f_n(t) - f_n(x_k)| < \epsilon/3$. Thus, for all $k > K$, $|f(t) - f(x_k)| < \epsilon$. This completes the proof.

Limits And Uniformly Convergent Series

In THEOREM 5.68, the word "continuous" can be replaced by "right continuous" or "left continuous." The next corollary follows from THEOREM 5.68 by replacing the function $f_n(x)$ by the partial sums $s_n(x)$.

5.69 COROLLARY If all of the functions in the sequence $a_k(x)$, $k = 0, 1, 2, \ldots$ are continuous at t and the series $a_0(x) + a_1(x) + \ldots$ converges uniformly on an interval containing t, then

$$\operatorname*{limit}_{x \rightarrow t} \sum_{k=0}^{\infty} a_k(x) = \sum_{k=0}^{\infty} a_k(t).$$

The most useful applications of COROLLARY 5.69 are to power series. Consider the Maclaurin series

$$\sum_{k=0}^{\infty} c_k x^k, \; |x| < R$$

where R is the radius of convergence. If $|t| < R$ then, by uniform convergence on $[-r, +r]$ where $|t| < r < R$, we have

$$\operatorname*{limit}_{x \rightarrow t} \sum_{k=0}^{\infty} c_k x^k = \sum_{k=0}^{\infty} c_k t^k.$$

If the series converges absolutely for x = R, then, by the M test, the series converges uniformly on $[-R,R]$. Thus,

$$\lim_{x \to R} \sum_{k=0}^{\infty} c_k x^k = \sum_{k=0}^{\infty} c_k R^k.$$

It is an interesting result, due to Abel, that the above equality is valid even if the series

$$\sum_{k=0}^{\infty} c_k R^k$$

converges only conditionally. This result can be proved from the ABEL'S TEST that we studied in 5.40 PROBLEMS BASED ON DIRICHLET'S TEST and ABEL'S TEST. Here is a sequence of steps leading to an interesting fact. You should be able to justify each step, the last of which uses Abel's limit theorem.

$$\frac{1}{1+t} = 1 - t + t^2 - t^3 + \dots \qquad |t| < 1$$

$$\frac{1}{1+t^2} = 1 - t + t^2 + t^4 - t^6 + \dots \qquad |t| < 1$$

$$\arctan(x) = \int_0^x \frac{dt}{1+t^2} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots \; |x| < 1$$

$$\frac{\pi}{4} = \lim_{x \to 1} \arctan(x) = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$$

The last identity, which can be written

$$\pi = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots\right)$$

is called Gregory's series for π. Try it on your computer!

5.70 EXERCISES

(1) Verify that all series marked "->" in the SERIES AND PRODUCTS section of the MATH TABLES in the Appendix are correct to at least four nonzero terms.

(2) Verify that all series marked "−>>" in the SERIES AND PRODUCTS section of the MATH TABLES in the Appendix are correct to at least four nonzero terms.

(3) Verify that all series marked "*" in the SERIES AND PRODUCTS section of the MATH TABLES in the Appendix are correct to at least four nonzero terms.

(4) Compute the TAYLOR SERIES expansions of e^x, sin(x), and cos(x) about $x = a$. Show your work clearly.

(5) Find the radius of convergence R of the following series from EXERCISE 5.42. Discuss convergence at R and $-R$ if $R < \infty$.

(a) $$\sum_{k=1}^{\infty} \frac{(k+1)(k+2)(k+3)}{k!} x^k$$

(b) $$\sum_{k=1}^{\infty} \frac{2^{k/2}}{k^2 + k + 1} x^k$$

(c) $$\sum_{k=1}^{\infty} \frac{k+1}{2k+3} x^k$$

(d) $$\sum_{k=0}^{\infty} \frac{k^5}{5^k} x^k$$

(e) $$\sum_{k=0}^{\infty} \frac{x^k}{k^2 - 150}$$

(f) $$\sum_{k=5}^{\infty} \frac{x^k}{2^k - k^2}$$

(g) $$\sum_{k=2}^{\infty} \frac{1}{(k^3 - k^2 - 1)^{1/2}} x^k$$

(h) $$\sum_{k=1}^{\infty} \frac{(k+1)^{1/2} - (k-1)^{1/2}}{k} x^k$$

(i) $\sum_{k=1}^{\infty} \frac{(-x)^k}{k}(1 + 2^{-2} + 3^{-2} + \cdots + k^{-2})$

(j) $\sum_{k=1}^{\infty} \frac{(-x)^k}{k}\left(\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{k}\right)$

(k) $\sum_{k=0}^{\infty} (-1)^k \operatorname{arccot}(k) x^k$

(l) $\sum_{k=0}^{\infty} \frac{\sin(k)}{|k - 99.5|} x^k$

(m) $\sum_{k=1}^{\infty} \frac{(-x)^k}{k} \cos(1/k)$

(n) $\sum_{k=0}^{\infty} \frac{-9k^2 - 5}{k^3 + 1} \sin(k) x^k$

(6) Find the radius of convergence R of the following series from VARIATIONS 5.44. Discuss convergence at R and $-R$ if $R < \infty$.

(a) $\sum_{k=1}^{\infty} \frac{(k-1)(k-2)(k-3)}{k!} x^k$

(b) $\sum_{k=1}^{\infty} \frac{2^{k/2}}{k^{200} + k + 1} x^k$

(c) $\sum_{k=1}^{\infty} \frac{k^2 + 1}{2k^2 + 3} x^k$

(d) $\sum_{k=0}^{\infty} \frac{k^{500}}{(1.01)^k} x^k$

(e) $\sum_{k=0}^{\infty} \frac{x^k}{k^{1.0001} - 150}$

(f) $\sum_{k=1}^{\infty} \frac{x^k}{2^k - 3k^{2000}}$

(g) $$\sum_{k=2}^{\infty} \frac{x^k}{(k^3 - k^2 - 1)^{1/4}}$$

(h) $$\sum_{k=1}^{\infty} \frac{(k+1)^{1/4} - (k-1)^{1/4}}{k} x^k$$

(i) $$\sum_{k=2}^{\infty} \frac{(-x)^k}{k} \left(\frac{1}{2(\ln(2))^2} + \frac{1}{3(\ln(3))^2} + \cdots + \frac{1}{k(\ln(k))^2} \right)$$

(j) $$\sum_{k=1}^{\infty} \frac{(-x)^k}{k} \left(\frac{(\ln(1))^2}{1} + \frac{(\ln(2))^2}{2} + \cdots + \frac{(\ln(k))^2}{k} \right)$$

(k) $$\sum_{k=1}^{\infty} (-x)^k \ln(k) \operatorname{arccot}(k)$$

(l) $$\sum_{k=0}^{\infty} \frac{\sin(k)}{\ln|k - 99.5|} x^k$$

(m) $$\sum_{k=1}^{\infty} \frac{(-x)^k}{k} \cos(k)$$

(n) $$\sum_{k=0}^{\infty} \frac{-9k^2 - 5}{k^3 + 1} \sin(k)\cos(k) x^k$$

(7) Find the radius of convergence R of the following series from VARIATIONS 5.45. Discuss convergence at R and $-R$ if $R < \infty$.

(a) $$\sum_{k=4}^{\infty} \frac{(k+1)(k+2)(k+3)}{k(k-1)(k-2)(k-3)} x^k$$

(b) $$\sum_{k=1}^{\infty} \frac{k^2}{2^{\ln(k)}} x^k$$

(c) $$\sum_{k=1}^{\infty} \frac{\ln(k) + (1/k)}{(\ln(k))^3 + 3} x^k$$

(d) $$\sum_{k=1}^{\infty} \frac{k^{\ln(k)}}{5^k} x^k$$

(e) $\sum_{k=1}^{\infty} \frac{x^k}{(\ln(k))^2 - 150}$

(f) $\sum_{k=5}^{\infty} \frac{x^k}{(\ln(\ln(k)))^k}$

(g) $\sum_{k=2}^{\infty} \frac{x^k}{(k^3 - k^2 - 1)^{1/2.99}}$

(h) $\sum_{k=1}^{\infty} \frac{(k+1)^{1/200} - (k-1)^{1/200}}{k} x^k$

(i) $\sum_{k=1}^{\infty} (-1)^k(e^{-1/k} - 1 + 1/k)x^k$

(j) $\sum_{k=1}^{\infty} (e^{-1/k} - 1 + 1/k)x^k$

(k) $\sum_{k=0}^{\infty} (-1)^k(\operatorname{arccot}(k))^{\beta}x^k, \ \beta < 1$

(l) $\sum_{k=0}^{\infty} \frac{\sin(k) - \cos(k)}{|k - 99.5|} x^k$

(m) $\sum_{k=1}^{\infty} \cos(k)\sin(1/k)x^k$

(n) $\sum_{k=0}^{\infty} \frac{-9k^2 - 5}{k^3 + 1} (\sin(k)\cos(k/2) + \cos(k)\sin(k/2))x^k$

(8) Find the radius of convergence R of the following series from VARIATIONS 5.46. Discuss convergence at R and $-R$ if $R < \infty$.

(a) $\sum_{k=1}^{\infty} \frac{k^k}{k!} x^k$

(b) $$\sum_{k=1}^{\infty} \frac{(k/4)^k}{k!} x^k$$

(c) $$\sum_{k=1}^{\infty} \frac{k!x^k}{k(k+1)\cdots(2k-1)}$$

(d) $$\sum_{k=1}^{\infty} \frac{(xk!)^{2k}}{(k^2)!}$$

(e) $$\sum_{k=1}^{\infty} \frac{(2k)!x^k}{k^2(k^2-1)\cdots(k^2-k+1)}$$

(f) $$\sum_{k=5}^{\infty} \left(\frac{1}{\ln(\ln(k))}\right)^{\ln(k)} x^k$$

(g) $$\sum_{k=5}^{\infty} x^k \left(\frac{1}{\ln(k)}\right)^{\ln(\ln(k))}$$

(h) $$\sum_{k=2}^{\infty} \frac{k^{\ln(k)}}{(\ln(k))^k} x^k$$

(i) $$\sum_{k=0}^{\infty} (-x)^k(\text{arccot}(k))^{\beta}, \ \beta > 1$$

(j) $$\sum_{k=0}^{\infty} \frac{\sin(k) - \cos(k)}{\ln(|k - 99.5|)} x^k$$

(k) $$\sum_{k=1}^{\infty} (\sin(k) - \sin(k + (1/k)))\, x^k$$

(l) $$\sum_{k=0}^{\infty} \ln(1 + (1/k))\sin(2k + \pi)x^k$$

(9) Consider the sequence of functions

$$a_k(x) = \frac{x}{1 + kx^2}.$$

Show that $a_k(x)$, $k = 0, 1, 2, \ldots$, tends to the zero function, $a(x) = 0$, uniformly on the domain $D = \{x: -\infty < x < \infty\}$. Discuss the convergence of the sequence of derivatives $a_k'(x)$ to $a'(x) = 0$.

(10) Consider the sequence of functions on [0, 1]

$$a_k(x) = \begin{cases} 0 & \text{if } 0 \le x < \dfrac{1}{k+1} \\ \sin(\pi/x) & \text{if } \dfrac{1}{k+1} \le x \le \dfrac{1}{k} \\ 0 & \text{if } \dfrac{1}{k} < x \le 1 \end{cases}$$

for $k = 1, 2, \ldots$. Does this sequence converge to a function on [0, 1]? Is convergence uniform? Prove or disprove your assertion.

(11) Consider the series

$$\sum_{k=1}^{\infty} a_k(x)$$

where $a_k(x)$ is as in **(10)**. Discuss the convergence of this series on [0, 1]. Does this series converge uniformly? Prove or disprove your assertion. Sketch some of the functions $s_n(x)$ in the sequence of partial sums of this series.

(12) Discuss the convergence of the series

$$s(x) = \sum_{k=0}^{\infty} \frac{1}{1 + k^2 x}.$$

For what x does this series converge absolutely? On what intervals does it converge uniformly? Prove or disprove your assertions.

(13) For what range of x can

(a) sin(x) be replaced by $x - x^3/6 + x^5/120$ with error less than .0001?

(b) cos(x) be replaced by $1 - x^2/2$ with error less than .001?

(c) e^x be replaced by $1 + x + x^2/2 + x^3/6$ with error less than .0001?

(d) $\ln(1 - x)$ be replaced by $-x - x^2/2 - x^3/3$ with error less than .01?

(14) Find the Maclaurin series for

(a) $\int_0^x \frac{1}{1 + t^3}\,dt$

(b) $\int_0^x \ln(1 + t^5)dt$

(c) $\int_0^x \frac{t^2}{(1 - t^2)^2}dt$

(15) Compute the Maclaurin series for $f(x) = (1 - x)^{-2}$ in the following three ways:

(a) Directly from the definition by computing derivatives $f^{(k)}(0)$, $k = 0, 1, 2, \ldots$.

(b) By differentiating the series for $(1 - x)^{-1}$ term by term. Justify your computations

(c) By direct multiplication of $(1 + x + x^2 + \ldots)$ times itself. Show that the coefficient of x^n in the series $(1 + x + x^2 + \ldots)^2$ is the number of pairs of nonnegative integers (i, j) with $i + j = n$. What is the corresponding interpretation of the coefficient of x^n in $(1 + x + x^2 + \ldots)^p$, $p > 2$, p an integer?

(16) Using Abel's limit theorem, discussed just after COROLLARY 5.69, show that $\ln(2) = 1 - (1/2) + (1/3) - (1/4) + \ldots$.

(17) By starting with the Maclaurin series

$$f(x) = a_0 + a_1x + a_2x^2 + \ldots$$

with the a_k unknown, substitute this series into the differential equation

$$x^2f'(x) - xf(x) = \sin(x).$$

Solve for the a_k and thus obtain a power series solution to this differential equation. REMARK: If $f(x)$ is a solution so is $f(x) + cx$ for any constant c.

(18) Obtain the Maclaurin series of the following functions in two different ways. The first way is directly, by computing the derivatives $f^{(k)}(0)$, and the second way is by direct multiplication of the Maclaurin series of the two factors.

(a) $e^{2x}\sin(x)$

(b) $\sin(x)\cos(2x)$

(c) $\sinh(x)\cosh(2x)$

(d) $\sin(x)\ln(1 + x)$

(19) Directly and by long division, obtain the first three nonzero terms of the Maclaurin series of the following functions:

(a) $\dfrac{\ln(1 + x)}{\cos(x)}$

(b) $\dfrac{\ln(1 + x)}{e^{\sin(x)}}$

(c) $\dfrac{e^x}{x^2 + x + 1}$

(d) $\dfrac{\sin(x)}{1 + x}$

(20) Directly and by composition, obtain the first four nonzero terms of the Maclaurin series of the following functions.

(a) $\tan(x^3 + 1)$

(b) $\cos(\sin(x))$

(c) $\cos(\ln(1 + x))$

(d) $\sin(1 + x + x^2)$