Top-down Calculus

Chapter 1

Linear Functions and Derivatives

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Preface

Top-down Calculus was developed in the 1980's for a summer session program to train high school teachers in San Diego County to teach calculus. These teachers had all taken calculus themselves, but they were wary of standing before a class and fielding questions – math anxiety of the "second kind." They knew about Newton, Leibniz, the falling apple, etc. What they didn't know was how to respond quickly when a student asked, "Hey teacher, how do I work this one?"

My approach in this book is to emphasize intuition and technique. The important chain rule is presented intuitively on page 11 (instead of page 100+ as in many standard calculus books). Exercises are presented as follows: prototypical exercise set, solutions and discussion, numerous exercise sets that are variations on the prototypes. My students (the teachers) were encouraged to work one or two of the variation exercise sets in detail and then scan the remaining variations noting the techniques required for each problem. The idea was that this "scanning" process would prepare them to deal with their own students' questions. They would be able to say with some confidence, "Well, Johnny, why don't you try this approach." At least this would buy time for them to think about the question more carefully.

Subsequent to the summer session program for high school teachers, I used this material for the one-quarter calculus course that I regularly taught in the Department of Mathematics at the University of California, San Diego. It seemed to work well for that purpose, but it is no competitor for the magnificent (but very expensive) standard calculus books. I also used this material for calculus taught in summer session. There, the concise nature of this material worked very well.

This chapter, Chapter 1, introduces the most important intuitive ideas needed to understand differential calculus.

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Chapter 1

LINEAR FUNCTIONS AND DERIVATIVES

Linear Functions Are The Foundation Of Calculus

A ''linear function'' is a function whose graph is a straight line. Almost every calculus book has a section or two on linear functions. We shall describe such functions in this chapter and leave it to the reader to supplement our discussion by reading in his or her precalculus textbook. Every linear function has an equation of the form $f(x) = ax + b$. In such an equation, x is called the ''independent variable'' or simply the ''variable.'' For example, $f(x) = 2x + 1$ is a linear function. The graph of this function and the graphs of two other linear functions are shown in FIGURE 1.1. The graph of the function $f(x) = 2x + 1$ is, by definition, the set of all points in the plane of the form $(x, 2x + 1)$. Thus (1, 3) and (2, 5) are points on the graph of $2x + 1$ but (1, 2) and (2, 4) are not on the graph of $2x + 1$. In the linear function $f(x) = ax + b$, the number a is called the ''slope of the function f'' and the number b is called the ''intercept of the function f'' or the ''vertical intercept of the function f.'' A discussion of these terms and the alternative ways of describing linear functions will be found in your algebra or precalculus textbook.

The Notation Of Calculus Demands Critical Attention

One of the major problems in learning calculus is the notation, which cynics say ''ranges from bad to horrible.'' We should begin now to think about

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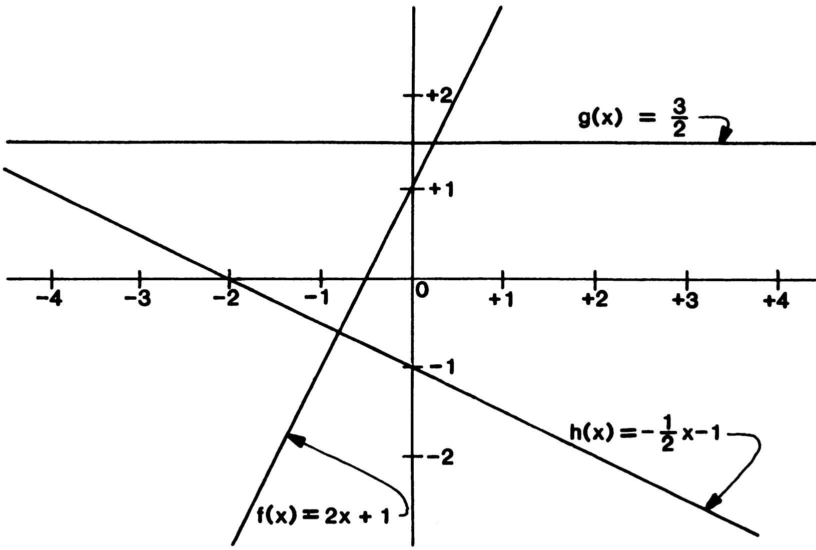
FIGURE 1.1 The Graphs of Three Linear Functions: f, g, and h



notation in a critical way. Take a look at FIGURE 1.1 again. The graph of the function $f(x) = 2x + 1$ is the "set of all points in the plane of the form $(x, 2x + 1)$." If you look in the previous sentence at the statement in quotes, there is no mention of the symbol f. Thus, if you had been asked to graph $p(x) = 2x + 1$ or $\beta(x) = 2x + 1$ you would have produced exactly the same graph! The only difference would be in how you would *refer* to the graph that you produced. If you have graphed $f(x) = 2x + 1$ you might say "Bill, put your finger on the graph of f." If you graphed $p(x) = 2x + 1$ or $\beta(x) = 2x + 1$ instead, then you would use p or β in making such a statement in place of f. One standard way of specifying a linear function is to use "y" rather than "f" as we have done. Of course, it makes no difference which letter you use except in referring to the linear function (as we have just noted). Thus, instead of $f(x) = 2x + 1$, one might write $y(x) = 2x + 1$. The notation "f(x)" tells us that f is the name of the function and x is the name of the variable. One could also write $f = 2x + 1$ or $y = 2x + 1$ and the function and variable names would be equally well specified. Oddly enough, one rarely sees $f = 2x + 1$, but $y = 2x + 1$ is very common. This sort of arbitrary notational tradition is quite common in calculus and is a source of confusion to the beginner.

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Playing The Envelope Game

There is a little game that the beginning calculus student can play that will help keep the question of notation in perspective. This game is called THE ENVELOPE GAME. We shall play it from time to time. For linear functions the game goes like this: Imagine you have a linear function in an envelope. Before you open up the envelope try and describe what it is that you will see on the inside. The object of the game is to list the various ways that a linear function might be described. One possibility is that the envelope contains a piece of graph paper with the graph of the linear function (a straight line, of course) on it. Another possibility is that one sees an equation of the form $y = ax + b$. Can you think of other possible contents of the envelope? What you should begin to understand from THE ENVELOPE GAME is the distinction between a mathematical object or concept and the manner used to specify or describe a particular instance of this object or concept. In the case at hand, the concept is that of a linear function which can be described in many different ways. A computer scientist might refer to these various ways of describing the same basic object as "different data structures."

Locally, Linear Functions Approximate Nonlinear Functions

THE PROPERTIES OF LINEAR FUNCTIONS ARE THE FOUNDATION OF ALL OF CALCULUS. There is a simple intuitive reason for this, which is shown in FIGURE 1.2. In FIGURE 1.2 we see the graph of a nonlinear function f. The function is nonlinear because its graph is not a straight line. Imagine that we choose three points on the graph of f and look at the curve under a microscope at these points. The three points we have selected are called A', B', and C' and are shown in FIGURE 1.2(a). The circles centered at these points represent the field of view of the microscope. In FIGURE 1.2(b) we see the view under the microscope centered at point B'. What we see is (essentially!) a straight line segment. This straight line segment defines a straight line as indicated by the dotted line of FIGURE 1.2(b). Like any straight line, it is the graph of a linear function of the form $y = ax + b$. By inspection, it appears that $a = -1/3$ in this case. What do you think b is for this straight line (a rough guess will do)? Actually, from the point of view of calculus, the value of b is not too important. It is the value of a that gets all of the attention. The value of a (in this case $-1/3$) is called *the derivative of f at B'*. Thus, to compute the derivative of a function f at a point B' on the graph of f one focuses a microscope at the point B',

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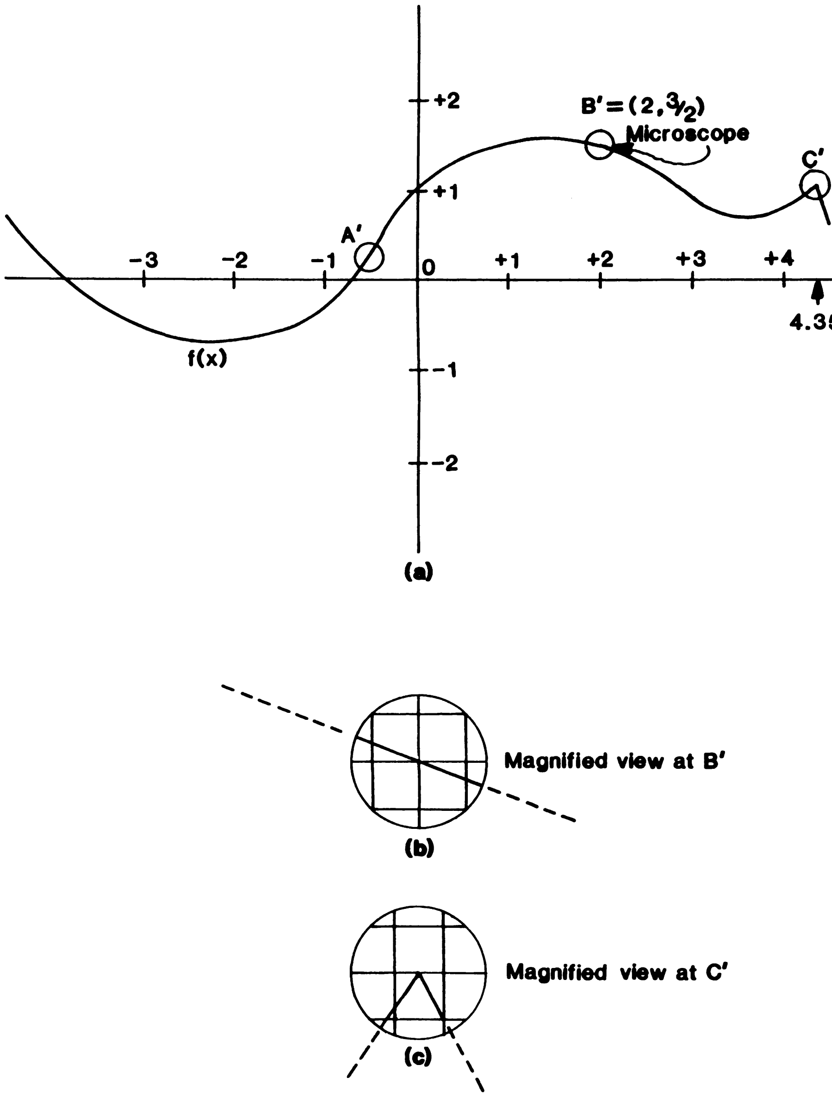
FIGURE 1.2 The Derivative of a Function at a Point



ups the magnification until the portion of the graph under the microscope looks like a straight line segment, and computes the slope of this line. The resulting number is *the derivative of f at B′*.

Let's play THE ENVELOPE GAME. Imagine that you have an envelope and inside it is the derivative of a function f at a point B′ on the graph of

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the function. Before opening up the envelope, describe what it is that you are going to see (Answer: a number). These ideas are summarized in SLOPPY DEFINITION 1.3.

The Intuitive Definition Of The Derivative

1.3 SLOPPY DEFINITION Let f be a function and let B′ be a point on the graph of f. Focus a microscope on the point B′ and increase the magnification until the portion of the graph in the field of view of the microscope looks like a straight line segment. The slope of this straight line segment is *the derivative of f at B′*.

A Function May Not Have A Derivative At Certain Points

One problem with SLOPPY DEFINITION 1.3 is shown at the point C′ in FIGURE 1.2(a). At this point, the graph of f has a sharp "spike" or "cusp." No matter how much we increase the magnification of the microscope, we never see a straight line segment! Thus, SLOPPY DEFINITION 1.3 doesn't work for such a point. We say, in this case, that "f does not have a derivative at C′." The functions that are studied in calculus have very few bad points such as C′ where the derivative does not exist. Some functions that naturally occur do have such points (for example, $f(x) = |x|$ has such a point at $x = 0$ and $f(x) = |x| + |x - 1|$ has two such points). Notice that the function shown in FIGURE 1.2 has infinitely many "good points" where the derivative exists but only one point C′ where the derivative does not exist (there may be more points not shown in FIGURE 1.2, but you get the idea!).

Compute The Derivative At Lots Of Points To Get The Derivative Function

We have been discussing the derivative of a function f *at a point* on the graph of f. Look now at FIGURE 1.4. There we see a function f(x). At each point on the graph of f(x) we have attempted to compute the derivative of f using SLOPPY DEFINITION 1.3. At the point (−4, +3.6), which is on the graph of f, we thought the derivative was about −0.4. At the point (−1, +1) on the graph of f we thought that the derivative was about −1.0. At (−7, +4.5) the derivative was 0, etc. Of course, there are infinitely many such points and we can't compute derivatives for all of them. We computed derivatives for a number of such points and then drew a smooth curve through them to obtain the graph of f′(x) shown in FIGURE 1.4. This new function is called *the derivative function of f*.

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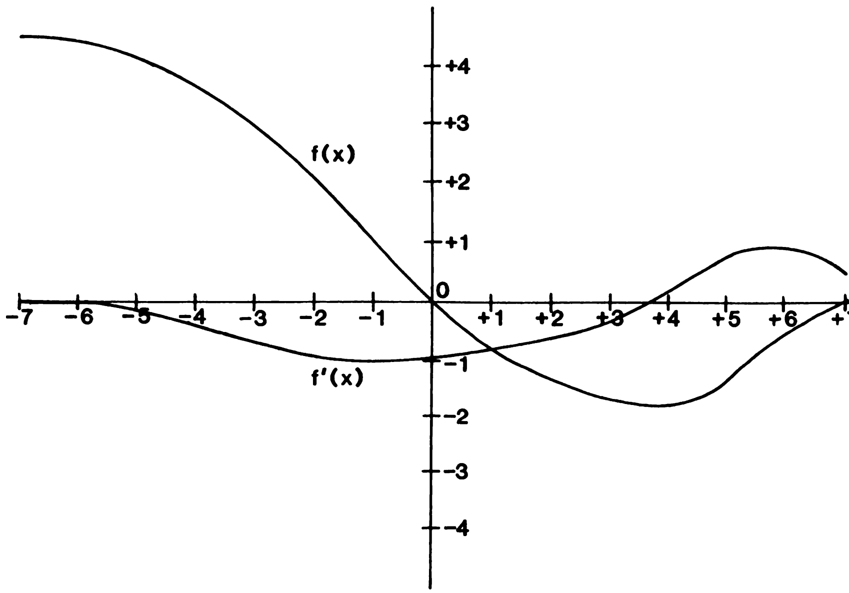
FIGURE 1.4 The Derivative Function f′ of a Function f



1.5 DEFINITION Let f be a function. For each point (x, f(x)) on the graph of f that has a derivative, compute that derivative and call it f′(x). The function f′(x) is called *the derivative function of f.*

Computing Derivative Functions Graphically

1.6 EXERCISES

(1) Draw the graph of the derivative functions f′, g′, and h′ for each of the three linear functions of FIGURE 1.1.

(2) Draw the graph of the derivative function f′ for the function f of FIGURE 1.2. The point C′ is a problem as there is no derivative at that point. Draw carefully what the graph of f′ looks like for points of the graph of f to the left of C′ (values of $x < 4.35$) and to the right of C′ (values of $x > 4.35$).

1.7 IMPORTANT PROPERTIES OF LINEAR FUNCTIONS We now take a look at some simple but extremely important properties of linear functions as they relate to calculus. To emphasize when we are talking about a *linear* function we shall use a "tilde" over the symbol representing that function. Thus $\tilde{f}$ will be a linear function.

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Suppose that $\tilde{f}(x) = ax + b$ and $\tilde{g}(x) = cx + d$ are two linear functions. Define $\tilde{s}(x) = \tilde{f}(x) + \tilde{g}(x)$ to be the *sum* of $\tilde{f}$ and $\tilde{g}$. Thus $\tilde{s}(x) = (a + c)x + (b + d)$. The important thing to notice here is that THE SLOPE OF THE SUM OF TWO LINEAR FUNCTIONS IS THE SUM OF THEIR SLOPES. It is evident from the calculation that we just did that the sum of two linear functions is again a linear function. This is not the case for the product of linear functions!

If a linear function is multiplied by any real number r, then we again obtain a linear function. For example, $r\tilde{f}(x) = rax + rb$. Note that the slope of the new linear function is r times the slope of the function $\tilde{f}$. Thus we have that THE SLOPE OF A REAL NUMBER TIMES A LINEAR FUNCTION IS THAT REAL NUMBER TIMES ITS SLOPE.

For Linear Functions, The Slope Of A Composition Is The Product Of The Slopes

The above two properties of linear functions are really pretty simple. We now consider a much deeper property of linear functions. Understanding this property *well* will be the key to a good grade in calculus!

Adding linear functions produces *another linear function* whose slope is the *sum* of the slopes of the two original linear functions. What sort of operation on linear functions produces *another linear function* whose slope is the *product* of the slopes of the two original linear functions? Taking products of linear functions won't do the job. For example, $(ax + b)(cx + d) = acx^2 + (ad + bc)x + bd$, which is not even linear (it's quadratic). It turns out that the operation that we need is *composition of functions*. The operation of composition of functions is not restricted to linear functions but we shall start with that case as it is particularly easy to follow. Let $\tilde{f}$ and $\tilde{g}$ be as in the previous paragraph. The *composition* of $\tilde{f}$ and $\tilde{g}$ is the function $\tilde{f}(\tilde{g}(x))$ obtained by replacing each occurrence of the variable x in $\tilde{f}(x)$ by the entire expression $\tilde{g}(x)$. Thus, if $\tilde{f}(x) = ax + b$ and $\tilde{g}(x) = cx + d$ then $\tilde{f}(\tilde{g}(x)) = a(cx + d) + b = acx + (ad + b)$. From this calculation we see that the composition of two linear functions is again a linear function and THE SLOPE OF THE COMPOSITION OF TWO LINEAR FUNCTIONS IS THE PRODUCT OF THEIR SLOPES. Let us summarize these important properties of linear functions:

(1) IF A REAL NUMBER IS MULTIPLIED TIMES A LINEAR FUNCTION, THEN THE NEW FUNCTION IS LINEAR AND ITS SLOPE IS THE PRODUCT OF THE REAL NUMBER AND THE SLOPE OF THE ORIGINAL FUNCTION.

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(2) THE SUM OF TWO LINEAR FUNCTIONS IS A LINEAR FUNCTION. THE SLOPE OF THE NEW FUNCTION IS THE SUM OF THE SLOPES OF THE ORIGINAL FUNCTIONS.

(3) THE COMPOSITION OF TWO LINEAR FUNCTIONS IS A LINEAR FUNCTION. THE SLOPE OF THE NEW FUNCTION IS THE PRODUCT OF THE SLOPES OF THE ORIGINAL FUNCTIONS.

We now must take a look at the meaning of composition of functions f and g where f and g may not be linear. Consider DEFINITION 1.8.

But We Can Compose Nonlinear Functions Too

1.8 DEFINITION Let f and g be functions (real valued). Define a function h whose value h(x) at any real number x is gotten by first evaluating g at x to obtain the real number g(x) and then evaluating f at g(x) to obtain f(g(x)). The function h is called the *composition of f and g*. We write h(x) = f(g(x)).

In calculus we deal primarily with functions whose "domain" and "range" are real numbers. These functions are called *real valued* and can be graphed using the standard horizontal and vertical real number lines (as in FIGURES 1.1 and 1.2, for example). Thus, when we form the composition of two real valued functions we obtain another real valued function.

It is now time to play THE ENVELOPE GAME with DEFINITION 1.8. Suppose we have an envelope and inside it are two functions f and g and their composition h(x) = f(g(x)). What are we going to see when we open the envelope? One possibility is that f and g and h are described by "formulas" or "closed expressions." Another possibility is that f and g and hence h are given as graphs. As an example of the formula type description, we might have $f(x) = 2x^3 + 5x^2 - 3x + 2$ and $g(x) = \sin(x) + \sqrt{x}$. Then we find that

$$f(g(x)) = 2(\sin(x) + \sqrt{x})^3 + 5(\sin(x) + \sqrt{x})^2 - 3(\sin(x) + \sqrt{x}) + 2.$$

As the above expression shows, the composition of two functions f and g given as formulas or closed expressions can be done just as with linear functions by replacing each occurrence of x in the expression for f(x) by the whole expression for g(x). There is a problem that might occur here. Suppose $f(x) = \sqrt{x}$ and $g(x) = x^3$. Let h be the composition of f and g. If we try to evaluate $h(-1)$ then we first form $g(-1) = -1$ and then try to evaluate $f(-1)$, which is the square root of -1. Thus, $h(-1)$ is not defined as a real number. In general, this sort of thing happens frequently. There will be real

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numbers x for which g(x) is defined but h(x) is not defined. This would be the case with the composition g(f(x)) using the g and f of the previous paragraph (used there to compute f(g(x))). You should easily be able to find values of x for which this function g(f(x)) is not defined. To find *all* such values is a little more work! Some exercises are given at the end of the chapter for the reader to practice these ideas. To consider the case where the two functions f and g being composed are given graphically, look at the *very important* example shown in FIGURE 1.9.

Composing Functions Graphically

In FIGURE 1.9 we see three functions f, g, and h given graphically. The function h is the composition of f and g: h(x) = f(g(x)). Computing the composition of functions given in graphical form can be a rather tedious process. In principle, one must compute f(g(x)) at each point x. Thus, for x = 3, we first compute g(3) by looking at the graph of g. It seems that g(3) = 2. We now compute f(g(3)) = f(2), which is about −1. Thus, by definition, h(3) = −1. If we do that for enough points then we can draw the graph of h as has been done in FIGURE 1.9(c). In FIGURE 1.9(a) a microscope has been placed at the point A′ = (3, g(3)) = (3, 2) on the graph of g. The straight line segment (or nearly so) that appears in the field of view (indicated by the circle) has been extended to obtain a straight line $\tilde{g}(x) = (1/3)x + 1$. This means, by SLOPPY DEFINITION 1.3, that slope($\tilde{g}$) = 1/3 is g′(3). Similarly, in FIGURE 1.9(c) a microscope has been put at the point B′ = (g(3), f(g(3))) = (2, −1). The line segment in the field of view at B′ has been extended to a straight line $\tilde{f}(x) = (-3/2)x + 2$.

Composing Linear Approximations To Functions

Suppose that two students, Larry (for Linear) and Nancy (for Nonlinear) are given the task of composing functions from FIGURE 1.9. Larry is going to compose $\tilde{f}$ and $\tilde{g}$ to obtain the function $\tilde{h}$ of FIGURE 1.9(c). Nancy is going to compose f and g to obtain the function h of FIGURE 1.9(c). Of course, their work is going to be quite different for most values of x. Note, however, that the graphs of $\tilde{g}$ and g coincide (essentially) in the circle about A′ and the graphs of $\tilde{f}$ and f coincide (essentially) in the circle about B′. Thus when Larry and Nancy are dealing with values in these circled regions they will essentially get the same answer for the composition as is shown in the circled region about C′ in FIGURE 1.9(c).

Larry, being a good student, has learned IMPORTANT PROPERTIES 1.7 (in particular number 3). Thus, he knows that the slope of $\tilde{f}$ times the slope

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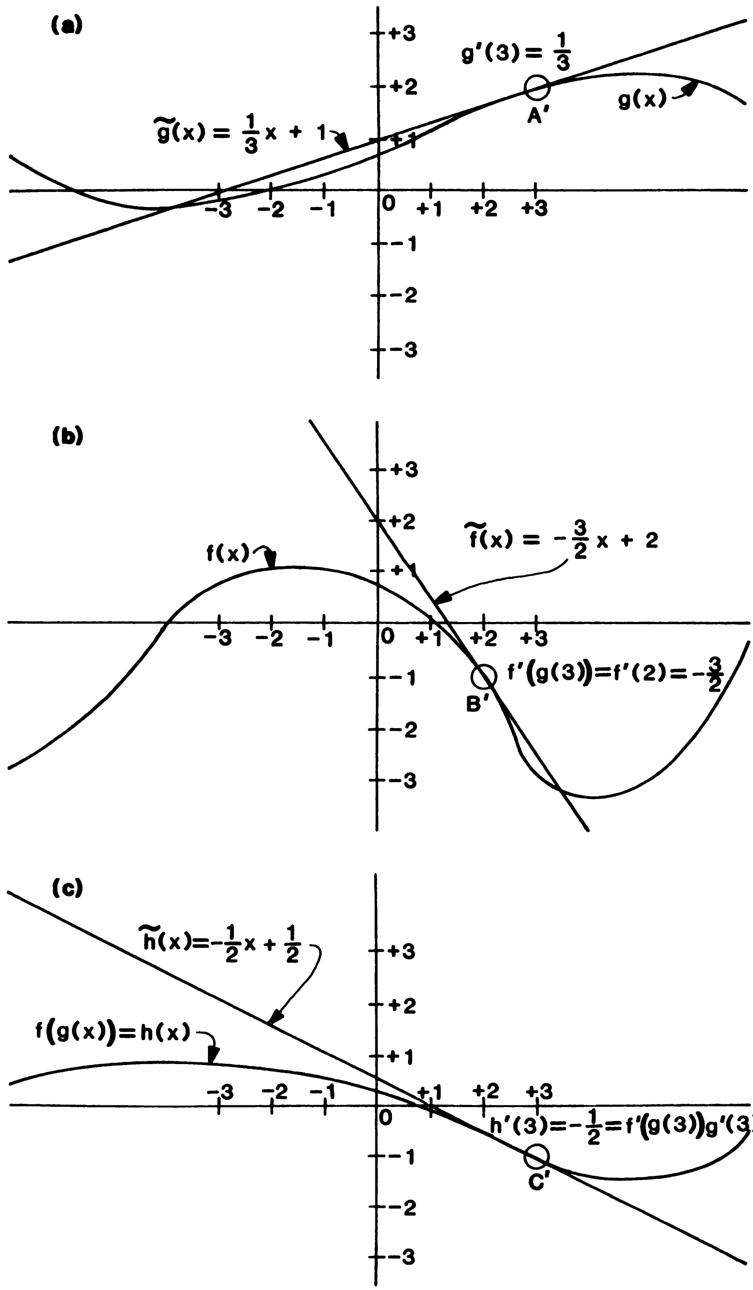
FIGURE 1.9 Graphical Composition and the Chain Rule



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of $\tilde{g}$ is the slope of $\tilde{h}$ (in this case $-3/2$ times $1/3$ equals $-1/2$). Nancy, also a good student, has learned SLOPPY DEFINITION 1.3, which she applies to the points A′, B′, and C′. She concludes correctly that the slope of $\tilde{g}$ is the derivative $g'(3)$ of g at the point A′, the slope of $\tilde{f}$ is the derivative $f'(g(3)) = f'(2)$ of f at the point B′, and the slope of $\tilde{h}$ is the derivative $h'(3)$ of h at the point C′. Putting their observations together, Larry and Nancy conclude that $h'(3) = f'(g(3))g'(3)$.

The same process would have worked for any x, not just $x = 3$ (the functions would have to be defined at x and g(x), of course, and have derivatives). Thus we would discover that $h'(x) = f'(g(x))g'(x)$. This observation is called *the chain rule* or *the composite function rule* and is without doubt the single most important fact that must be learned and thoroughly understood by the beginning calculus student. We state this rule in IMPORTANT THEOREM 1.10.

The Most Important Concept To Master . . .

1.10 IMPORTANT THEOREM (THE CHAIN RULE) Let f(x) and g(x) be functions and let $h(x) = f(g(x))$ be the composition of f and g. Then, the derivative $h'(x)$ is equal to $f'(g(x))g'(x)$.

Many beginning calculus students would be much happier if IMPORTANT THEOREM 1.10 stated $h'(x) = f'(x)g'(x)$ but, alas, this is not true in general. It is not true for $x = 3$ in FIGURE 1.9, as the reader should verify.

Linearity Of The Derivative: $(rf(x) + sg(x))' = rf'(x) + sg'(x)$

As we have just seen, IMPORTANT PROPERTY 1.7(3) of linear functions gives rise to IMPORTANT THEOREM 1.10 on derivatives. What rules of derivatives follow from IMPORTANT PROPERTIES 1.7(1) and (2)? Fortunately, these rules are so simple that most beginning calculus students apply them automatically. There are times, however, when these rules need to be articulated clearly in solving a problem so we shall now discuss them briefly.

Take a look at FIGURE 1.11. There we see the graphs of functions f(x), g(x), and h(x). These functions have been constructed so that $h(x) = f(x) + g(x)$. As with FIGURE 1.9, microscopes have been placed on these graphs at points A′, B′, and C′. These points all correspond to $x = +1$. Thus $A' = (1, f(1)) = (1, 3)$, $B' = (1, g(1)) = (1, -2)$, and $C' = (1, h(1)) = (1, 1)$. In the microscope, each curve looks like a straight line segment. These straight line segments are extended to give the lines $\tilde{f}$, $\tilde{g}$, and $\tilde{h}$ as shown in

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FIGURE 1.11 Derivative of a Sum Is the Sum of the Derivatives

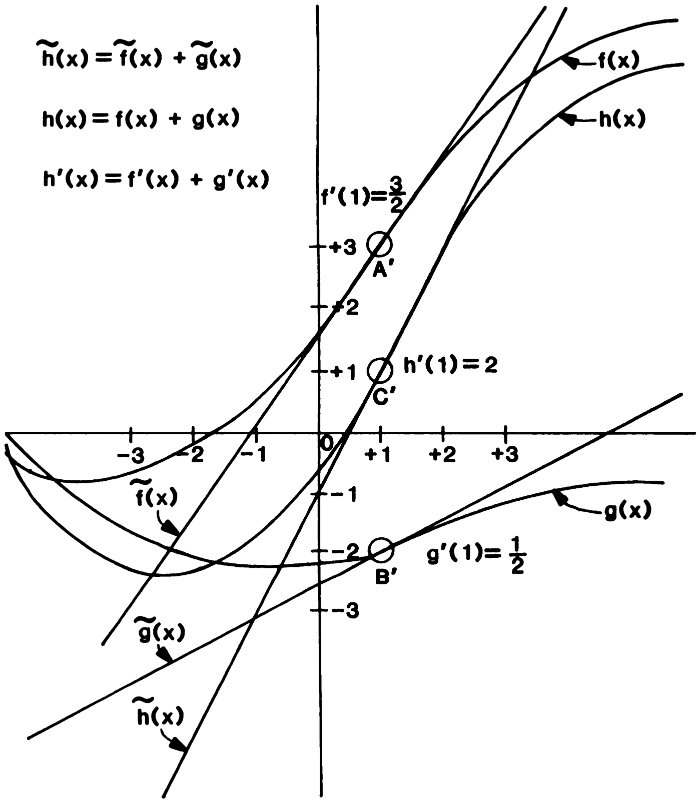

FIGURE 1.11. Clearly, $\tilde{h} = \tilde{f} + \tilde{g}$. By IMPORTANT PROPERTY 1.7(2), slope($\tilde{h}$) = slope($\tilde{f}$) + slope($\tilde{g}$) and hence by SLOPPY DEFINITION 1.3, $h'(1) = f'(1) + g'(1)$, as can be seen in the example of FIGURE 1.1. Instead of using $x = 1$ we could have used any x in this argument (provided that the relevant derivatives were defined) and we would obtain $h'(x) = f'(x) + g'(x)$. This rule states that *the derivative of a sum* (in this case $h'(x)$) *is the sum of the derivatives* (in this case $f'(x) + g'(x)$). The rule corresponding to IMPORTANT PROPERTY 1.7(1) is even easier: if $h(x) = rf(x)$ where h and f are functions and r is a real number, then $h'(x) = rf'(x)$. The reader should explain this rule graphically as was done for the other two rules in FIGURE 1.9 and FIGURE 1.11 respectively. We state these three rules in RULES FOR DERIVATIVES 1.12.

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1.12 RULES FOR DERIVATIVES

(1) CONSTANT MULTIPLE RULE: If f(x) is a function and r is a constant (i.e., a real number) and $h(x) = rf(x)$ then $h'(x) = rf'(x)$.

(2) SUM RULE: If f(x) and g(x) are functions and $h(x) = f(x) + g(x)$ then $h'(x) = f'(x) + g'(x)$.

(3) CHAIN RULE: If f(x) and g(x) are functions and $h(x) = f(g(x))$ then $h'(x) = f'(g(x))g'(x)$.

The Envelope Game Again. . .

Understanding the RULES FOR DERIVATIVES 1.12 has been the basic object of this chapter. The next chapter will be devoted to the problem of "computing derivatives." To begin our thinking about this problem, let's play THE ENVELOPE GAME once again. Suppose we are given a function f(x) graphically, as in FIGURE 1.4. We are given an envelope and inside the envelope is the derivative function f'(x). What is it that we are going to see when we open the envelope? Most likely it will be another graph, just as in FIGURE 1.4. If this were all that was involved in the study of derivatives, calculus would be a trivial subject. Given any function f(x), we would draw its graph and compute f'(x) as in FIGURE 1.4. This would work for any function. For example, it would work for the function $f(x) = x^2$. You may have already had enough experience with calculus to know, however, that if your calculus instructor asked you to find the derivative of $f(x) = x^2$ and your answer was a graph, he or she would probably "flip out." The expected answer would be "$f'(x) = 2x$." In other words, given a function f(x) as a formula or "closed expression" and an envelope containing f'(x), one would expect to open the envelope and find another formula or "closed expression." So given $f(x) = x^2$ the envelope should contain $f'(x) = 2x$.

If $f(x) = x^2$ Then $f'(x) = 2x$. Here's Why. . .

Every calculus book proves that if $f(x) = x^2$ then $f'(x) = 2x$. To understand how this interesting fact relates to SLOPPY DEFINITION 1.3, take a look at FIGURE 1.13. In FIGURE 1.13(a) we see a portion of the graph of $f(x) = x^2$. In the spirit of SLOPPY DEFINITION 1.3, we have put our microscope at the point $A' = (1, 1)$ on the graph of f. The slope of the straight line segment in the field of view of the microscope seems to be about 2. If $f'(x) = 2x$ is the correct formula then this is as it should be as $f'(1) = 2$.

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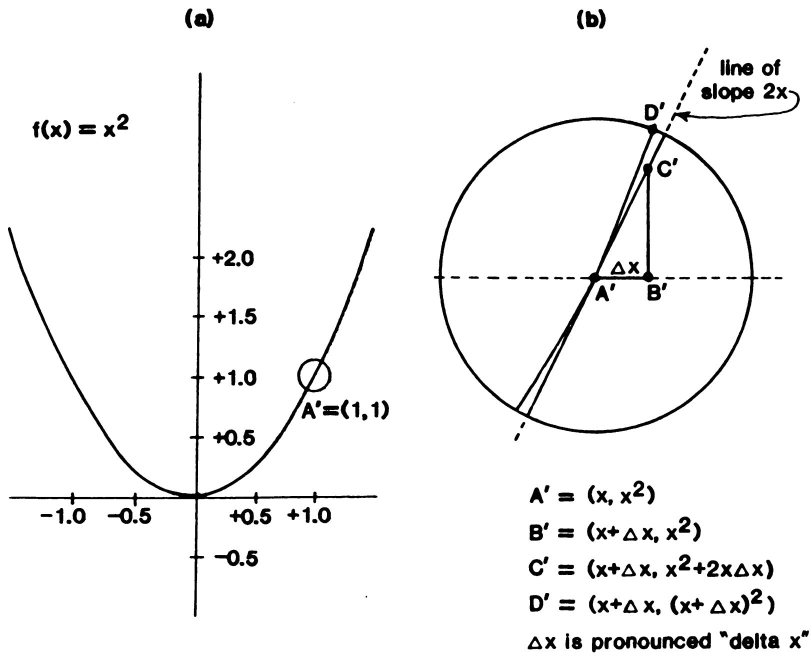
FIGURE 1.13 Derivative of $f(x) = x^2$



The Square Of A Small Number Is Even Smaller

In FIGURE 1.13(b) we are taking a more careful look at the field of view of the microscope at the point A′. To be a little more general, we are looking at $A' = (x, x^2)$ (instead of just $x = 1$ as shown in FIGURE 1.13(a)) where x is some value near 1. In FIGURE 1.13(b) the line joining A′ to C′ is a straight line segment of slope 2x. The distance from A′ to B′ is Δx. The symbol "Δx" is used in calculus to stand for a "small number which is to be added to x to change it slightly." The line joining A′ to D′ is a part of the graph of $f(x) = x^2$ and is not, of course, exactly a straight line segment.

To see how far off from being straight this latter line is, let's compute the distance the point D′ is above the point C′. The second coordinate of C′ is $x^2 + 2x\Delta x$ and the second coordinate of D′ is $(x + \Delta x)^2 = x^2 + 2x\Delta x + (\Delta x)^2$. The difference between these two coordinates is $(\Delta x)^2$, which is the height of D′ above C′. We have in mind that Δx is small, say .001. In this case, $(\Delta x)^2 = .000001$, is much smaller still. The ratio of the second of these two quantities to the first is $\Delta x = .001$. Referring to FIGURE 1.13(b),

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we have shown that the ratio of the distance between D′ and C′ to the distance between A′ and B′ is Δx. Thus, this ratio goes to zero as Δx goes to zero. This is the analytic statement of the fact that the smaller the field of view (which is roughly Δx) the more the part of the graph in the field of view appears to be a straight line.

The Difference Quotient: $\Delta f = f(x + \Delta x) - f(x)$ Over Δx

Look again at FIGURE 1.13(b). If the line from A′ to D′ was a straight line segment then its slope would be $\dfrac{f(x + \Delta x) - f(x)}{\Delta x} = \dfrac{(x + \Delta x)^2 - x^2}{\Delta x} = 2x + \Delta x$. The quantity $\dfrac{f(x + \Delta x) - f(x)}{\Delta x}$ is called the "difference quotient of f with respect to Δx." As we have just seen, the difference quotient is almost $f'(x)$ and the smaller Δx is the closer this difference quotient is to $f'(x)$. Mathematicians would express this fact by saying that "the limit of $\dfrac{f(x + \Delta x) - f(x)}{\Delta x}$ as Δx tends to zero is $f'(x)$." This statement assumes that f has a derivative at x. This important idea is summarized in ERUDITE OBSERVATION 1.14.

1.14 ERUDITE OBSERVATION If f is any function that has a derivative at x then the difference quotient $\dfrac{f(x + \Delta x) - f(x)}{\Delta x}$ is very close to $f'(x)$ for small enough values of Δx. In other words, the limit of $\dfrac{f(x + \Delta x) - f(x)}{\Delta x}$ as Δx tends to zero is $f'(x)$. In symbols we may write

$$\operatorname*{limit}_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = f'(x).$$

Another common notation for ERUDITE OBSERVATION 1.14 is to define $\Delta f = f(x + \Delta x) - f(x)$. We then may write $\operatorname*{limit}_{\Delta x \to 0} \dfrac{\Delta f}{\Delta x} = f'(x)$. In the case where $f(x) = x^2$ we found that $\dfrac{\Delta f}{\Delta x} = 2x + \Delta x$ and thus, $f'(x) = 2x$.

In the next chapter we shall be concerned with techniques for computing derivatives. Most of what the beginning calculus student has to learn is precisely "techniques of computation." The major difficulty in this task is the profusion of notation brought about by the many different subjects to

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which calculus has been applied. We close this chapter by reviewing the ideas we have discussed thus far through the study of a number of specific examples.

1.15 EXAMPLES OF RULES FOR DERIVATIVES 1.12

$\frac{d}{dx}f$ Is $\frac{df}{dx}$ Is f′(x) In Differential Notation

(1) We now know how to compute the derivative function of $f(x) = x^2$. The answer is $f'(x) = 2x$. Another standard notation for the derivative function $f'(x)$ is $\frac{df}{dx}$. Thus if $f(x) = x^2$ then $\frac{df}{dx} = 2x$. The notation $f'(x)$ tells us that f' is a function of x. In the notation $\frac{df}{dx}$ it is the x in dx that tells us that the variable is x. Sometimes one sees $\frac{df}{dx}(x)$ or $\frac{df(x)}{dx}$ if one wants to emphasize that the variable is x. Another way of saying that the derivative of x^2 is 2x is to write $(x^2)' = 2x$ or $\frac{d}{dx}x^2 = 2x$. Unfortunately, one must get used to all of these notations!

Constant Functions Have Zero Derivative Functions

(2) A function $f(x) = r$ where r is a number is called a "constant function." Thus $f(x) = 2$ is a constant function. The graph of $f(x) = 2$ is a line parallel to and two units above the horizontal axis. The function g(x) of FIGURE 1.1 is a constant function with value 3/2.

Beginning students sometimes confuse constant functions with constants (i.e., numbers). There is an important difference. The function $f(x) = 2$ is a rule that assigns to every real number x the number 2 in the same sense that $f(x) = x^2$ assigns to every x the value x^2. The graph of $f(x) = 2$ is the set of all pairs (x, 2). The graph of $g(x) = 3/2$ of FIGURE 1.1 is the set of all pairs (x, 3/2) as shown. The numbers 2 or 3/2 are not the same as the functions $f(x) = 2$ or $g(x) = 3/2$. Any constant function, such as $f(x) = 2$, has slope zero at every point of its graph. Thus $f'(x) = 0$ for such a function.

In practice, the possibility of being seriously confused by the difference between constants and constant functions is very small. If one sees a statement such as $\frac{d}{dx}(2) = 0$ then one knows that the constant function with value 2 is being differentiated (i.e., having its derivative taken) and its derivative func-

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tion is the constant function with value 0. Another way of saying the same thing is $(2)' = 0$. No matter how large the value of the constant function, its derivative function is still the zero function. Thus, $(1{,}000{,}000)' = 0$.

The Derivative Of A Linear Function Is Its Slope

(3) In EXERCISE 1.6, the reader was asked to compute the derivative functions of the three linear functions of FIGURE 1.1. The answers are $f'(x) = 2$, $g'(x) = 0$, and $h'(x) = -1/2$. In general, if $f(x) = ax + b$ is a linear function then $f'(x) = a$ is the constant function with value equal to the slope of f(x). Thus, $\frac{d}{dx}(23x + 45) = 23$, $(-12x - 124)' = -12$, and $\frac{d}{dx}x = 1$. This latter result can also be written $(x)' = 1$ and, probably because it is so simple, is sometimes a source of confusion to the beginner.

The Derivative Of A Sum Is. . .

(4) We now take a look at RULES FOR DERIVATIVES 1.12. Applying 1.12(1), we compute $(2x^2)' = 2(x^2)' = 2(2x) = 4x$. In our alternative notation, $\frac{d}{dx}(2x^2) = 2\frac{d}{dx}(x^2) = 2(2x) = 4x$. In a similar fashion, $(45x^2)' = 90x$, $(-4x^2)' = -8x$. Applying 1.12(2) we can compute $(2x^2 + 4x)' = (2x^2)' + (4x)'$ which, by 1.12(1), is $2(x^2)' + 4(x)' = 4x + 4$. In our alternative notation we would have $\frac{d}{dx}(2x^2 + 4x) = \frac{d}{dx}(2x^2) + \frac{d}{dx}(4x) = 4x + 4$. As stated, RULE 1.12 applies to the sum of two functions, but it is obviously valid for 3, 4, or any finite sum of functions. Thus $(e(x) + f(x) + g(x))' = e'(x) + (f(x) + g(x))' = e'(x) + f'(x) + g'(x)$. As an example, $(5x^2 + 6x + 9)' = (5x^2)' + (6x)' + (9)' = 10x + 6 + 0 = 10x + 6$.

Using The Chain Rule Requires Some Guesswork

(5) Rules 1.12(1) and 1.12(2) illustrated in example (4) above are easily mastered by the beginning student and usually applied correctly with little thought involved. The CHAIN RULE, 1.12(3), is a different matter! As we have already stated, it is the most important rule of calculus, at least in the beginning. The CHAIN RULE concerns functions of the form f(g(x)) which are compositions of two functions f(x) and g(x). The CHAIN RULE is then used to compute the derivative $(f(g(x)))'$ of this function. The CHAIN RULE requires that you compute first $f'(x)$ and then compose this with g(x) to get

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$f'(g(x))$. This is then multiplied times $g'(x)$ to get $f'(g(x))\, g'(x)$, which is the correct answer.

That sounds easy enough. For example, let $f(x) = x^2$ and let $g(x) = 9x^2 - 6x + 4$. Then $f(g(x)) = (9x^2 - 6x + 4)^2$. We compute that $f'(x) = 2x$ and $g'(x) = 18x - 6$ and hence $f'(g(x))g'(x) = 2(9x^2 - 6x + 4)(18x - 6)$. In practice, there is an additional complication: $f(g(x))$ may be given but not $f(x)$ and $g(x)$. Suppose we are asked to differentiate $(3x^2 + 3x + 3)^{2/3}$. We think we would like to use the CHAIN RULE, but what are $f(x)$ and $g(x)$? The answer is that we must guess what they are! In this case we could take $f(x) = x^2$ and $g(x) = (3x^2 + 3x + 3)^{1/3}$ or we could take $f(x) = x^{2/3}$ and $g(x) = 3x^2 + 3x + 3$. Both work to give the expression $(3x^2 + 3x + 3)^{2/3}$ for $f(g(x))$. In terms of applying the CHAIN RULE, the latter is a much better choice. To apply the CHAIN RULE in this case we need to know that $(x^r)' = rx^{r-1}$ for any real number r (not just $r = 2$ as we have already shown). This formula will be derived in the next chapter and can just be accepted for now. Using this formula, we see that $f'(x) = (2/3)x^{-1/3}$. Obviously by now, $g'(x) = 6x + 3$ and so

$$f'(g(x))g'(x) = (2/3)\,(3x^2 + 3x + 3)^{-1/3}\,(6x + 3),$$

which is the correct derivative. We shall continue worrying about the CHAIN RULE in the next example.

The Chain Rule In Differential Notation

(6) Suppose we are going to apply the CHAIN RULE to $f(g(x))$ where $f(x) = x^3$. We would first find $f'(x) = 3x^2$ and then substitute $g(x)$ for x to obtain $f'(g(x)) = 3(g(x))^2$, which we may write more simply as $f'(g) = 3g^2$. If instead of writing $f(x) = x^3$ we had written $f(g) = g^3$ and differentiated with respect to g to get $\frac{df}{dg} = 3g^2$ we would have obtained the same result directly. For this reason, $f'(g(x))g'(x)$ may be written $\frac{df}{dg}\frac{dg}{dx}$. The CHAIN RULE may now be stated $\frac{df}{dx} = \frac{df}{dg}\frac{dg}{dx}$.

But We Still Must Guess A Lot. . .

(7) In applying the CHAIN RULE in the form $\frac{df}{dx} = \frac{df}{dg}\frac{dg}{dx}$ we are faced with the same difficulties as in example (5) above. Suppose we are asked to find the derivative of $(2x^2 + 1)^3 + 8(2x^2 + 1)^2 + 6(2x^2 + 1) + 3$.

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We would like to think of this expression as f(g(x)) for some f and g, but what are f and g? Again we must guess. In this case it looks like a natural choice for g is the function $2x^2 + 1$ which occurs throughout this expression. With this choice for g, $f = g^3 + 8g^2 + 6g + 3$. We compute $\frac{df}{dg} = 3g^2 + 16g + 6$ and $\frac{dg}{dx} = 4x$. Thus, $\frac{df}{dx} = (3g^2 + 16g + 6)4x = (3(2x^2 + 1)^2 + 16(2x^2 + 1) + 6)4x$. Of course, we could use any two distinct symbols for f and g. If we had used v and w instead the formula would be $\frac{dv}{dx} = \frac{dv}{dw}\frac{dw}{dx}$.

$\frac{d}{dx} g^r = rg^{r-1} \frac{dg}{dx}$

(8) As we remarked in example (5) above, $(x^r)' = rx^{r-1}$ for any real number r. We shall derive this result in the next chapter. This means that for any function g(x), $((g(x))^r)' = r(g(x))^{r-1}g'(x)$. This follows from the CHAIN RULE with $f(x) = x^r$. In our alternative notation

$$\frac{d}{dx}(g(x))^r = r(g(x))^{r-1}\frac{d}{dx}g(x).$$

It is fun to apply this rule over and over again to see what complicated expressions one can derive. For example, take $g(x) = x^3 + 1$ and $r = .20$. Then $\frac{d}{dx}(x^3 + 1)^{.2} = .2(x^3 + 1)^{-.8}3x^2$. Let $g(x) = ((x^3 + 1)^{.2} + 1)$ and $r = .5$. Then $\frac{d}{dx}((x^3 + 1)^{.2} + 1)^{.5} = .5((x^3 + 1)^{.2} + 1)^{-.5}.2(x^3+1)^{-.8}3x^2$.

Now we can repeat this process a few more times and you will amaze your friends with the complex functions you can differentiate (i.e., find the derivative function of). It is at this point that the beginning student, amazed by the complexity of the calculations, forgets completely what the subject is about! This is the time to go back and look at FIGURES 1.2 and 1.4. Also, rethink the CHAIN RULE and what it means in terms of FIGURE 1.9. No matter how complicated the expressions, these are the ideas that underlie the formulas we have been deriving in these examples.

Differentiating Different Expressions For The Same Function

(9) As our final example, we shall differentiate the two expressions $(x^2 + x^3)^2$ and $x^4 + 2x^5 + x^6$. Actually, these two expressions represent the same function, as the second is what we get by computing the square indicated

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by the first. Since these two expressions represent the same function, the expressions obtained by differentiating them must also represent the same function. Using the CHAIN RULE, we compute $\frac{d}{dx}(x^2 + x^3)^2 = 2(x^2 + x^3)(2x + 3x^2)$ and, using the rule for differentiating sums, we obtain $\frac{d}{dx}(x^4 + 2x^5 + x^6) = 4x^3 + 10x^4 + 6x^5$. These two expressions look different at first glance but we know they are the same as functions (i.e., give the same value for each value of x) because they are just different descriptions of the derivative of the same function. In fact, multiplying the terms of the first expression gives the second.

This idea can be used to prove a very useful general identity for derivatives. To do this, replace x^2 by f(x) and x^3 by g(x) where f(x) and g(x) are any two functions that can be differentiated (''differentiable functions''). We shall try and compute $\frac{d}{dx}(f(x) + g(x))^2$ in two different ways as above. First, using the CHAIN RULE we obtain

$$\frac{d}{dx}(f(x) + g(x))^2 = 2(f(x) + g(x))(f'(x) + g'(x))$$

$$= 2f(x)f'(x) + 2(f'(x)g(x) + f(x)g'(x)) + 2g(x)g'(x).$$

Second, by first computing the square, we obtain

$$\frac{d}{dx}((f(x))^2 + 2f(x)g(x) + (g(x))^2)$$

$$= 2f(x)f'(x) + 2(f(x)g(x))' + 2g(x)g'(x).$$

The Product Rule

By setting these two expressions equal to each other and canceling common terms we obtain the important ''product rule''

$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$$

In our derivation of this result, we mixed our two notations for the derivative. This is fine as long as the meaning is clear. In our alternative notation, our new formula for the derivative of a product of two functions becomes

$$\frac{d}{dx}(f(x)g(x)) = \left(\frac{d}{dx}f(x)\right)g(x) + f(x)\left(\frac{d}{dx}g(x)\right).$$

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We shall have much more to say about this important formula in the next chapter.

So Now Memorize It: $(fg)' = f'g + fg'$

We now give some exercises. Following the exercises we give the solutions. Try to work each exercise first without looking at the solution. If you do look at the solution to an exercise, *immediately* make up on your own a variation of that exercise and work your variation. Finally, we give a complete set of variations of these exercises for you to practice with. In the Appendix, Supplementary Reading and Exercises, you will find related reading assignments and exercises from various standard calculus books. If you have access to one of these books, you should test yourself by looking at this material.

Read The Previous Paragraph Again And Resolve To Do It!

1.16 EXERCISES

Some Routine Work With Compositions

(1) Find the compositions indicated below. In the cases indicated, specify the values for which the composite function is defined.

(a) Find $h(x) = f(g(x))$ where $f(x) = 2x^3 + 3$ and $g(x) = (-x - 1)^3$.

(b) Find $h(x) = f(g(x))$ where $f(t) = 2t^3 + 3$ and $g(x) = (-x - 1)^3$.

(c) Find $u(v(x))$ where $u = 3v^4 + 2v^3 + 3v + 6$ and $v(x) = -x^2 + 1$.

(d) Find $p(q(x))$ where $p = \dfrac{1}{2y^3 + 3}$ and $q = x^{1/2}$.

(e) Find $h(z) = f(g(z))$ where $f(g) = \dfrac{g + 1}{g - 1}$ and $g(z) = \sqrt{z}$. For what values of z is h(z) defined?

(f) Find $h(k(y))$ and $k(h(y))$ where $h(y) = y^2$ and $k(y) = y^{1/2}$. For what values of y are these functions defined?

(g) Find $h(k(y))$ and $k(h(y))$ where $h(y) = y^3$ and $k(y) = y^{1/3}$. For what values of y are these functions defined?

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Now Some Guesswork. . .

(2) Find functions f(g) and g(x) such that $h(x) = f(g(x))$ in each of the following cases. There are many possible correct answers in each case but generally only one "natural" choice.

(a) $h(x) = (x^2 + 1)^{1/3} + (x^2 + 1)^{-1/3}$

(b) $h(x) = (x + 1)^3 + x^2 + 2x + 1$

(c) $h(x) = \dfrac{x^2 + 2x}{(x + 1)^5}$

(d) $h(x) = \dfrac{x^2 + 2x + 5}{x^2 + 2x + 6}$. For this case, try $g(x) = x^2 + 2x + 6$, $g(x) = x^2 + 2x + 5$, and $g(x) = x + 1$.

There's More Than One Way To Do It. . .

(3) In each of the following cases a function h(x) is given. For each of the specified functions g(x), find f(g) such that $f(g(x)) = h(x)$.

(a) For $h(x) = \dfrac{1}{(x^{8/5} - x^2)^{1/2}}$ find f(g) when $g(x) = x^{8/5} - x^2$ and $g(x) = x^{1/5}$.

(b) For $h(x) = \dfrac{1}{\sqrt{9 - 4x^2}}$ find f(g) when $g(x) = 9 - 4x^2$ and $g(x) = (9 - 4x^2)^{1/2}$, and $g(x) = 2x/3$.

(c) For $h(x) = \dfrac{1}{\sqrt{1 - 9x^2}}$ find f(g) when $g(x) = 1 - 9x^2$ and $g(x) = 3x$.

(d) For $h(x) = (16 - 2x^2)^{-1/2}$ find f(g) when $g(x) = 16 - 2x^2$ and $g(x) = x/\sqrt{8}$.

(e) For $h(x) = \dfrac{1}{(x^{4/3} + x^{2/3})^2}$ find f(g) when $g(x) = x^{4/3} + x^{2/3}$, $g(x) = x^{2/3}$, and $g(x) = x^{2/3} + 1/2$.

(f) For $h(x) = \dfrac{1}{(x - x^{4/7})^5}$ find f(g) when $g(x) = x - x^{4/7}$ and $g(x) = x^{1/7}$.

(g) For $h(x) = \dfrac{4x + x^{1/2} + 1}{4x + x^{1/2} + 5}$ find f(g) when $g(x) = 4x + x^{1/2} + 5$ and $g(x) = 2x^{1/2} + 1/4$.

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Now Some Guesswork. . .

(4) For each of the following choices of $f(g)$ and $g(x)$, sketch the graph of $h(x) = f(g(x))$. Be reasonably accurate but don't nitpick!

(a) $f(g) = g^{1/2}$ and $g(x) = \sin(x)$

(b) $f(g) = g^{1/3}$ and $g(x) = \sin(x)$

(c) $f(g) = |g|$ and $g(x) = \sin(x)$

At Last, Some Derivatives To Try

(5) Compute the following derivatives:

(a) $\frac{d}{ds}s =$

(b) $\frac{d}{dx}\left(x^{1/2} + \frac{1}{(2x + 1)^{1/3}}\right) =$

(c) $\frac{d}{dx}(x^3 + 5x + 9)^{10} =$

(d) $\frac{d}{dx}\left(\frac{6}{x^3 + 2x + 1}\right)^{10} =$

(e) If $f(x) = \frac{1}{\sqrt{x^3 + 9x^5}}$ then $f'(x) =$

(f) $\frac{d}{dt}(t^{3/2} + t^{-3/2}) =$

(g) Find $F'(2)$ if $F(x) = 8x^3 + 3x^{-1}$.

(h) If $D(x) = (A(x) + B(x) + C(x))^2$ find $D'(2)$ if $A(2) = B(2) = 1$, $C(2) = 2$, $A'(2) = B'(2) = 1/2$, and $C'(2) = 3$.

(i) Find the equation of the line tangent to $y = x^3 + 3x^2 + 3$ at $x = 1$.

(j) Find the equation of the line normal to the curve $y = x^{1/3}$ at $x = 8$.

(k) For $f(x) = \frac{1}{2x - 1}$ find $f'(x)$ and the second derivative $f''(x)$.

(l) For $f(x)$ as in (i) above, find the "third derivative" $\frac{d^3}{dx^3}f(x)$.

(m) Find $\left[\frac{d}{dt}w(t^3 + t^2 + t + 1)\right]_{t=1}$ if $w'(4) = 6$.

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After Reading A Solution, Rework The Exercise, Changed Slightly

1.17 SOLUTIONS TO EXERCISE 1.16

(1) **(a)** For each occurrence of x in $f(x) = 2x^3 + 3$ we substitute the expression $(-x - 1)^3$. Thus we write $h(x) = 2((-x - 1)^3)^3 + 3 = 2(-x - 1)^9 + 3$.

(b) For each occurrence of t in $f(t) = 2t^3 + 3$ we substitute the expression $(-x - 1)^3$. Obviously, we get the same answer as in (a) above. The choice of symbol for the variable in the f function does not affect the resulting function f(g(x)).

(c) We get $u(v(x)) = 3(-x^2 + 1)^4 + 2(-x^2 + 1)^3 + 3(-x^2 + 1) + 6$. This function is a polynomial of degree 8 in x and could, if we do some tedious algebra, be written as a sum of powers of x with integer coefficients. There will be situations where this sort of algebra will be important, but not here! The answer is best left in this form in the absence of any direct motivation for doing otherwise.

(d) We have $p(q(x)) = \dfrac{1}{2(x^{1/2})^3 + 3} = \dfrac{1}{2x^{3/2} + 3}$.

(e) We substitute $\sqrt{z}$ for each occurrence of g in $\dfrac{g + 1}{g - 1}$ to obtain

$f(g(z)) = \dfrac{\sqrt{z} + 1}{\sqrt{z} - 1}$. The function $\sqrt{z}$ is defined (as a function from real numbers to real numbers) only for z a nonnegative real number. If $z = 1$ then the denominator of f(g(z)) is zero. Thus f(g(z)) is defined for all nonnegative real numbers z except $z = 1$.

Compositional Inverses Will Haunt You Later

(f) First, $h(k(y)) = (y^{1/2})^2 = y$. In reverse order we find $k(h(y)) = (y^2)^{1/2} = y$. The function $i(y) = y$ is the "identity" function (it does nothing to y). The function i(y) is the linear function with slope 1 passing through the origin. In general, functions h(y) and k(y) such that $h(k(y)) = k(h(y)) = y$ for all y are called "compositional inverses of each other." We must be a little bit careful about the statement "for all y" in this definition. In our example, the function $h(y) = y^2$ is defined for all y but the function $k(y) = y^{1/2}$ is defined only for non-negative real numbers (we are ignoring complex numbers at this point). Thus, the composition k(h(y)) is defined for all real numbers but the composition h(k(y)) is defined only for nonnegative real numbers. Tech-

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nically, the statement $h(k(y)) = k(h(y))$ can be made only for nonnegative real numbers and hence h and k are compositional inverses for all *nonnegative* real numbers. In general, when we say that two functions h and k are compositional inverses we mean that $h(k(y)) = k(h(y)) = y$ for all y in some specified common domain of definition of h and k. In our example this "common domain of definition" is the set of all nonnegative real numbers.

(g) As in (f) above, $h(k(z)) = k(h(z)) = z$. In this case, however, this relation is defined for all real numbers z because $z^{1/3}$ is defined for all real numbers and, of course, so is z^3. In general, if p is an odd integer then z^p and $z^{1/p}$ are compositional inverses for all z. If p is even (and nonzero) they are compositional inverses for all nonnegative z.

(2) **(a)** $f(g) = g^{1/3} + g^{-1/3}$ and $g(x) = x^2 + 1$ is the most natural choice. Also $f(g) = g^{2/3} + g^{-2/3}$ and $g(x) = (x^2 + 1)^{1/2}$, $f(g) = g + g^{-1}$ and $g(x) = (x^2 + 1)^{1/3}$, etc. will work. There are infinitely many possibilities. . .

(b) $h(x) = (x + 1)^3 + (x + 1)^2$ so we may take $f(g) = g^3 + g^2$ and $g(x) = x + 1$ as the most natural choice.

Remember This Trick! Completing The Square

(c) The numerator in h(x) is the expression $x^2 + 2x$. This expression is the sum of the first two terms of $(x + 1)^2 = x^2 + 2x + 1$ and thus can be written as $x^2 + 2x = (x + 1)^2 - 1$. This little trick is called "completing the square." It can be applied to any expression $ax^2 + bx$ by observing that $(\sqrt{a}x + b/2\sqrt{a})^2 = ax^2 + bx + b^2/4a$ and hence that

$$ax^2 + bx = (\sqrt{a}x + b/2\sqrt{a})^2 - b^2/4a.$$

You should learn this trick thoroughly and try a number of examples until you feel comfortable with it! Applying it to our immediate problem gives $f(g) = \dfrac{g^2 - 1}{g^5} = g^{-3} - g^{-5}$ and $g(x) = x + 1$.

(d) Using the "complete the square" trick described in (c) above, we replace $x^2 + 2x$ by $(x + 1)^2 - 1$ in both the numerator and denominator of h(x). Doing this we see that, in the case where $g(x) = x + 1$, we have $f(g) = \dfrac{g^2 + 4}{g^2 + 5}$. If $g(x) = x^2 + 2x + 6$ then $f(g) = (g - 1)/g$ or $1 - g^{-1}$. If $g(x) = x^2 + 2x + 5$ then $f(g) = g/(g + 1)$.

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Integration By Substitution Is What We'll Call It Later. . .

(3) As a general remark about this problem, the CHAIN RULE is very important in both differential calculus (that's what we are studying now) and integral calculus (we'll study that a little later). The tricks for writing a given function h(x) as f(g(x)) are a little different in these two subjects. That's the motivation for the different choices for g(x) in these problems.

(a) For $g(x) = x^{8/5} - x^2$, $f(g) = g^{-1/2}$. For $g(x) = x^{1/5}$, $f(g) = g^{-4}(1 - g^2)^{-1/2}$.

(b) The notation $\frac{1}{\sqrt{9 - 4x^2}}$ is a bad one and should be replaced by $\frac{1}{(9 - 4x^2)^{1/2}}$ or simply $(9 - 4x^2)^{-1/2}$. When $g(x) = 9 - 4x^2$ then $f(g) = g^{-1/2}$. When $g(x) = (9 - 4x^2)^{1/2}$ then $f(g) = g^{-1}$. When $g(x) = 2x/3$ then $x = 3g/2$ and thus $(9 - 4x^2)^{-1/2} = (9 - 4(3g/2)^2)^{-1/2} = (9 - 9g^2)^{-1/2} = \frac{1}{3}(1 - g^2)^{-1/2} = f(g)$. This is an important trick in "integral calculus."

(c) When $g(x) = 1 - 9x^2$ then $f(g) = g^{-1/2}$. If $g(x) = 3x$ then $x = g/3$ so $(1 - 9x^2)^{-1/2} = (1 - 9(g/3)^2)^{-1/2} = (1 - g^2)^{-1/2}$. Compare this latter "change of variable" with that of (b) above.

(d) When $g(x) = 16 - 2x^2$ then $f(g) = g^{-1/2}$. When $g(x) = x/\sqrt{8}$ then $x = \sqrt{8}g$ and $(16 - 2x^2)^{-1/2} = (16 - 2(\sqrt{8}g)^2)^{-1/2} = \frac{1}{4}(1 - g^2)^{-1/2}$.

(e) When $g(x) = x^{4/3} + x^{2/3}$ then $f(g) = g^{-2}$. When $g(x) = x^{2/3}$ then $x = g^{3/2}$ and $f(g) = (g^2 + g)^{-2}$. Completing the square we find that $f(g) = ((g + 1/2)^2 - 1/4)^2$. If $g = x^{2/3} + 1/2$ then $f(g) = (g^2 - 1/4)^{-2}$.

(f) When $g(x) = x - x^{4/7}$ then $f(g) = g^{-5}$. When $g(x) = x^{1/7}$ then $f(g) = (g^7 - g^4)^{-5} = g^{-20}(g^3 - 1)^{-5}$.

(g) If $g(x) = 4x + x^{1/2} + 5$ then $f(g) = \frac{g - 4}{g} = 1 - 4g^{-1}$. For the second part, note that $4x + x^{1/2} = (2x^{1/2} + 1/4)^2 - 1/16$. Here we are again completing the square using $x^{1/2}$ as the variable. Thus for $g(x) = 2x^{1/2} + 1/4$ we get $f(g) = \frac{g^2 + 15/16}{g^2 + 79/16}$.

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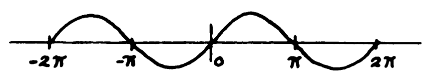
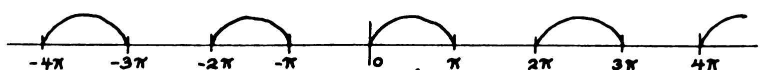
When Graphing, Do A Rough Approximation First

(4)

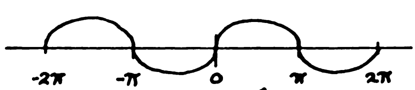


sin(x)

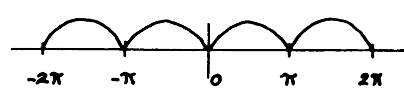


$(\sin(x))^{1/2}$

$|\sin(x)|$



$(\sin(x))^{1/3}$



(5) Be sure to try the VARIATIONS for additional practice in taking derivatives.

(a) $\frac{d}{ds}s = \frac{d}{dt}t = \frac{d}{dy}y = \frac{d}{dw}w = \frac{d}{dx}x = \ldots = 1.$

(b) $\frac{d}{dx}\left(x^{1/2} + \frac{1}{(2x + 1)^{1/3}}\right) = \frac{d}{dx}x^{1/2} + \frac{d}{dx}(2x + 1)^{-1/3} =$ $(1/2)x^{-1/2} + (-1/3)\ (2x + 1)^{-4/3}(2) = (1/2)x^{-1/2} - (2/3)(2x + 1)^{-4/3}$. The second term was computed using the CHAIN RULE applied to $f(g(x))$ with $f(g) = g^{-1/3}$ and $g(x) = 2x + 1$ (and hence $g'(x) = 2$).

(c) Apply the CHAIN RULE to $f(g(x))$ where $f(g) = g^{10}$ and $g(x) = x^3 + 5x + 9$. The answer is $10(x^3 + 5x + 9)^9(3x^2 + 5)$.

(d) We can write this as $(6^{10})\frac{d}{dx}(x^3 + 2x + 1)^{-10}$ and use the CHAIN RULE for $f(g(x))$ with $f(g) = g^{-10}$ and $g(x) = x^3 + 2x + 1$ to obtain $6^{10}(-10)(x^3 + 2x + 1)^{-11}(3x^2 + 2))$. Another, and probably more common, way a calculus student would attempt to work this problem

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is to apply the CHAIN RULE with $g(x) = \dfrac{6}{x^3 + 2x + 1}$ and $f(g) = g^{10}$ to get

$$10\left(\frac{6}{x^3 + 2x + 1}\right)^9 \frac{d}{dx}\left(\frac{6}{x^3 + 2x + 1}\right).$$

The latter derivative $\dfrac{d}{dx}\left(\dfrac{6}{x^3 + 2x + 1}\right)$ would then be computed with a second application of the CHAIN RULE. The reader should carry this out and verify that the answer is the same as the one just given.

(e) We should first get rid of the awkward square root notation and write this function as $(x^3 + 9x^5)^{-1/2}$. Use the CHAIN RULE with $g(x) = x^3 + 9x^5$ and $f(g) = g^{-1/2}$ to get $(-1/2)(x^3 + 9x^5)^{-3/2}(3x^2 + 45x^4)$.

(f) The answer is $(3/2)t^{1/2} + (-3/2)t^{-5/2}$.

(g) $F'(x) = 24x^2 + 3(-1)x^{-2} = 24x^2 - \dfrac{3}{x^2}$ and thus $F'(2) = 96 - 3/4 = 95.25$.

(h) We again use the CHAIN RULE with $f(g) = g^2$ and $g(x) = A(x) + B(x) + C(x)$ so that $D(x) = f(g(x))$. Thus, $D'(x) = 2(A(x) + B(x) + C(x))(A'(x) + B'(x) + C'(x))$ and $D'(2) = 2(1 + 1 + 2)(1/2 + 1/2 + 3) = 32$.

(i) If you look at FIGURE 1.9(a), the line $\tilde{g}$ is tangent to the function g at the point (3, 2) on the graph of g. In general, if g(x) is a function that has a derivative at the point $P = (x, g(x))$ then the *tangent to g at P* is the unique straight line passing through P and having slope $g'(x)$. In this problem, we are asked to find the tangent to $y(x) = x^3 + 3x^2 + 3$ at the point $x = 1$. This is a typical way of stating this sort of problem in which the statement "at $x = 1$" really means "at the point $(1, y(1)) = (1, 7)$." We must find the equation of the straight line passing through the point (1, 7) and having slope $y'(1) = 9$. This is the line $y = 9x - 2$.

(j) The normal line to a curve at a point P is the line passing through P and perpendicular to the tangent line to the curve at P. In this problem, $P = (8, 2)$. The tangent line to $y = x^{1/3}$ at P has slope $y'(8) = (1/3)8^{-2/3} = 1/12$ and its equation is given by $y = (1/12)x + 4/3$. You should know that the slope of the line normal to a line of slope m is $-1/m$. Thus the normal line we are looking for has slope -12 and

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passes through the point (8, 2). Its equation is $y = -12x + 98$, which is the answer to this problem.

(k) Write $f(x) = (2x - 1)^{-1}$. We find that $f'(x) = (-1)(2x - 1)^{-2}(2x - 1)' = -2(2x - 1)^{-2}$. The function $f'(x)$ can again be differentiated. The first derivative of $f'(x)$ is called the *second derivative* of $f(x)$ and is denoted by $f''(x)$ or by $\frac{d^2}{dx^2}$. In this problem, we compute $f''(x) = (-2(2x - 1)^{-2})' = (-2)(-2)(2x - 1)^{-3}(2x - 1)' = 8(2x - 1)^{-3}$.

(l) The third derivative of $f(x)$ is the derivative of $f''(x)$. The third derivative is denoted by $f'''(x)$ or by $\frac{d^3}{dx^3}$. In this problem we compute $(8(2x - 1)^{-3})' = (8)(-3)(2x - 1)^{-4}(2x - 1)' = -48(2x - 1)^{-4}$. In general, the n^{th} derivative of $f(x)$ is obtained by differentiating $f(x)$ n times. The n^{th} derivative is denoted by $f^{(n)}(x)$ or by $\frac{d^n}{dx^n}f(x)$. Can you give a formula for $\frac{d^n}{dx^n}(2x + 1)^{-1}$? In general, it will happen that a function with a first derivative at a point may not have an n^{th} derivative at that point for some n. For example, $f(x) = x^{4/3}$ has first derivative function $f'(x) = (4/3)x^{1/3}$ and second derivative $f''(x) = (4/9)x^{-2/3}$. At $x = 0$ we have $f'(0) = 0$ but $f''(0)$ is not defined.

(m) This problem illustrates the "bracket notation" for the two-step process of *first* computing a derivative function and *second* evaluating that function at a certain value. The notation $\left[\frac{d}{dx}f(x)\right]_{x=a}$ means first compute $f'(x)$ and then evaluate $f'(a)$. Some students like to do it the other way around by first finding $f(a)$ and then computing the derivative with respect to x of the constant $f(a)$. The answer is of course always zero! This is not the way to go. In our particular problem, we compute

$$\frac{d}{dt}w(t^3 + t^2 + t + 1) = w'(t^3 + t^2 + t + 1)(t^3 + t^2 + t + 1)'$$

$$= w'(t^3 + t^2 + t + 1)(3t^2 + 2t + 1).$$

Substituting $t = 1$ this becomes $w'(4)(6) = 6w'(4)$. We are using the CHAIN RULE without knowing explicitly what w is (sometimes this is a very useful trick). We could go no further except for the fact that we have (conveniently!) been given that $w'(4) = 6$ so the final answer is 36.

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Do You See How The Variations Parallel The Original Problems?

1.18 VARIATIONS ON EXERCISE 1.16

(1) Find the compositions indicated below. In the cases indicated, specify the values for which the composite function is defined.

(a) Find $h(x) = f(g(x))$ where $f(x) = 2x^5 - 3$ and $g(x) = (-x + 1)^{1/3}$.

(b) Find $h(x) = f(g(x))$ where $f(t) = 2t^5 - 3$ and $g(x) = (-x + 1)^3$.

(c) Find $u(v(x))$ where $u = 2v^5 + 2v^3 + 3v + 9$ and $v(x) = (-x^2 - 1)^{-1}$.

(d) Find $p(q(x))$ where $p = \dfrac{1}{2y^{-3} - 3}$ and $q = x^{-1/2}$.

(e) Find $h(z) = f(g(z))$ where $f(g) = \dfrac{g^3 + 1}{g^3 - 1}$ and $g(z) = \sqrt{z}$. For what values of z is $h(z)$ defined?

(f) Find $h(k(y))$ and $k(h(y))$ where $h(y) = y^6$ and $k(y) = y^{1/6}$. For what values of y are these functions defined?

(g) Find $h(k(y))$ and $k(h(y))$ where $h(y) = y^5$ and $k(y) = y^{1/5}$. For what values of y are these functions defined?

(2) Find functions $f(g)$ and $g(x)$ such that $h(x) = f(g(x))$ in each of the following cases. There are many possible correct answers in each case but generally only one "natural" choice.

(a) $h(x) = (x^2 + 1)^{1/3} + (x^4 + 2x^2 + 1)^{-1/3}$

(b) $h(x) = (x^{1/2} + 1)^3 + x + 2x^{1/2} + 9$

(c) $h(x) = \dfrac{4x^2 + 4x + 9}{(2x + 1)^5}$

(d) $h(x) = \dfrac{x^3 + 3x^2 + 3x + 5}{x^2 + 2x + 6}$. For this case, try $g(x) = x + 1$.

(3) In each of the following cases a function $h(x)$ is given. For each of the specified functions $g(x)$, find $f(g)$ such that $f(g(x)) = h(x)$.

(a) For $h(x) = \dfrac{1}{(x^{3/2} - x^{2/3})^2}$ find $f(g)$ when $g(x) = x^{2/3}$ and $g(x) = x^{3/2}$.

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(b) For $h(x) = \dfrac{1}{\sqrt{4 - 9x^2}}$ when $g(x) = (4 - 9x^2)^{1/2}$, $g(x) = 3x$, and $g(x) = 3x/2$.

(c) For $h(x) = \dfrac{1}{\sqrt{1 - 8x^3}}$ find $f(g)$ where $g(x) = 1 - 8x^3$ and $g(x) = 2x$.

(d) For $h(x) = (10 - 2x^2)^{-1/2}$ find $f(g)$ when $g(x) = 10 - 2x^2$ and $g(x) = x/\sqrt{5}$.

(e) For $h(x) = \dfrac{1}{(x^{4/3} + x^{2/3} + 5/4)^2}$ find $f(g)$ when $g(x) = x^{4/3} + x^{2/3}$, $g(x) = x^{2/3}$, and $g(x) = x^{2/3} + 1/2$.

(f) For $h(x) = \dfrac{1}{(x - x^{4/3})^3}$ find $f(g)$ where $g(x) = x - x^{4/3}$ and $g(x) = x^{1/3}$.

(g) For $h(x) = \dfrac{4x + 2x^{1/2} + 1}{4x + 2x^{1/2} + 5}$ find $f(g)$ where $g(x) = 4x + 2x^{1/2} + 5$ and $g(x) = 2x^{1/2} + 1/2$.

(4) For each of the following choices of $f(g)$ and $g(x)$, sketch the graph of $h(x) = f(g(x))$.

(a) $f(g) = \sin(g)$ and $g(x) = 1/x$

(b) $f(g) = \sin(g)/g$ and $g(x) = 1/x$

(c) $f(g) = \tan(g)$ and $g(x) = 1/x$

(5) Take each of (a) through (m) of EXERCISE 1.16 (5) and vary it slightly. Change constants, exponents, and variable names. Write down the problems thus created and work them. Try some of your classmate's problems also.

1.19 VARIATIONS ON EXERCISE 1.16

(1) Find the compositions indicated below. In the cases indicated, specify the values for which the composition is defined.

(a) Find $h(x) = f(g(x))$ where $f(x) = 2x^2 + 4$ and $g(x) = (-x + 1)^5$.

(b) Find $h(x) = f(g(x))$ where $f(t) = 3t^2 - 2$ and $g(x) = (x - 1)^2$.

(c) Find $u(v(x))$ where $u(v) = 3v^5 - 2v^2 + v - 1$ and $v(x) = 2x^3 - 14$.

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(d) Find p(q(x)) where $p = \dfrac{1}{q^4 - q}$ and $q = x^{1/3}$.

(e) Find $h(z) = f(g(z))$ where $f(g) = \dfrac{g - 2}{g + 2}$ and $g(z) = \sqrt{z} - 3$.

For what values of z is h(z) defined?

(f) Find h(k(y)) and k(h(y)) where $h(y) = y^4$ and $k(y) = y^{1/4}$. For what values of y are these functions defined?

(g) Find h(k(y)) and k(h(y)) where $h(y) = y^{1/5}$ and $k(y) = y^5$. For what values of y are these functions defined?

(2) Find functions f(g) and g(x) such that $h(x) = f(g(x))$ in each of the following cases. There are many possible answers but usually one "natural" answer.

(a) $h(x) = (x - 1)^3 - 3(x - 1)^2 + \sqrt{x - 1} + 4$

(b) $h(x) = (x + 2)^2 + x^2 + 4x + 5$

(c) $h(x) = \dfrac{x^2 + 9x + 4}{(x + 3)^2}$

(d) $h(x) = \dfrac{x^2 - 1}{x^2 + 1}$. Try $g(x) = x^2$, $g(x) = x^2 - 1$, $g(x) = x^2 + 1$.

(3) In each of the following cases a function h(x) is given. For each of the specified functions g(x), find f(g) such that $f(g(x)) = h(x)$.

(a) For $h(x) = \sqrt{x^2 - x^3}$ find f(g) when $g(x) = x^2 - x^3$ and $g(x) = x^{1/4}$.

(b) For $h(x) = (25 - 4x^2)^3 - 4$ find f(g) when $g(x) = 25 - 4x^2$, and $g(x) = 2x/5$.

(c) For $h(x) = (1 - 8x^3)^{-1/2}$ find f(g) when $g(x) = 1 - 8x^3$, and $g(x) = x/2$.

(d) For $h(x) = 1/(x^{1/5} - x^{3/5})^4$ find f(g) when $g(x) = x^{1/5} - x^{3/5}$, $g(x) = x^{1/5}$, and $g(x) = x^{1/5} + 1$.

(e) For $h(x) = (3 - 2x^2)^{5/2}$ find f(g) when $g(x) = 3 - 2x^2$, and $g(x) = \sqrt{2}x/\sqrt{3}$.

(f) For $h(x) = (x^{2/9} - x + 1)^{-2}$ find f(g) when $g(x) = x^{2/9} - x$, and $g(x) = x^{1/9}$.

(g) For $h(x) = \dfrac{x + 3\sqrt{x} - 1}{x + 3\sqrt{x} + 4}$, find f(g) when $g(x) = x + 3\sqrt{x}$, and $g(x) = \sqrt{x} + 3/2$.

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(4) For each of the following choices of f(g) and g(x), sketch the graph of $h(x) = f(g(x))$. Be reasonably accurate, but don't worry about being perfect.

(a) $f(g) = g^2$, and $g(x) = \cos(x)$

(b) $f(g) = 2g + 1$ and $g(x) = \cos(x)$

(c) $f(g) = \sqrt{g}$ and $g(x) = \cos(x) + 1$

(5) Take each of (a) through (m) of EXERCISE 1.16 (5) and vary it slightly. Change constants, exponents, and variable names. Write down the problems thus created and work them. Try some of your classmate's problems also.

1.20 VARIATIONS ON EXERCISE 1.16

(1) Find the compositions indicated below. In the cases indicated, specify the values for which the composition is defined.

(a) Find $h(x) = f(g(x))$ where $f(x) = 3x^4 + 4x^2 + 1$ and $g(x) = x + 2$.

(b) Find $h(x) = f(g(x))$ where $f(x) = \dfrac{1}{x - 2}$ and $g(x) = x - 2$.

(c) Find $r(s(x))$ where $r(s) = \sqrt{s} - 4s + 1$ and $s(x) = 2x$.

(d) Find $m(n(x))$ where $m(n) = (n + 4)^3 - 1$ and $n(x) = 14x - 37$.

(e) Find $h(x) = f(g(x))$ where $f(x) = \dfrac{(x + 1)^2 - 1}{(x + 1)^2 + 1}$ and $g(x) = 2x^2 + 3$.

(f) Find $h(u) = f(g(u))$ where $f(g) = 2/(g - 3)$ and $g(u) = \sqrt{u + 2}$. For what values of u is h(u) defined?

(g) Find f(g(x)) and g(f(x)) where $f(x) = (x + 2)^{-2}$ and $g(x) = x^3 - 1$. For what values of x are f(g((x)) and g(f(x)) defined?

(h) Find f(g(x)) and g(f(x)) where $f(x) = \sqrt{x}$ and $g(x) = x^2 + x$. For what values of x are these functions defined?

(2) Find functions f(g) and g(x) such that $h(x) = f(g(x))$ in each of the following cases. There are many possible answers but generally only one "natural" choice.

(a) $h(x) = \sqrt{x^2 + 4x}$

(b) $h(x) = (x + 1)^3 - 3(x + 1)^2 + 4$

(c) $h(x) = (x - 2)^2 + 2(x^2 - 4x + 5)$

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(d) $h(x) = \frac{x^2 - 2x + 3}{x^2 - 2x + 5}$. Try $g(x) = x^2 - 2x + 3$, and $g(x) = (x - 1)$.

(3) In each of the following cases a function is given. For each of the specified functions g(x), find f(g) such that $f(g(x)) = h(x)$.

(a) For $h(x) = (1/x^{1/3} - 1/x^{2/3})^3$ find f(g) when $g(x) = 1/x^{1/3} - 1/x^{2/3}$, $g(x) = x^{1/3}$.

(b) For $h(x) = 1/\sqrt{x^2 + 2}$ find f(g) when $g(x) = \sqrt{x^2 + 2}$, $g(x) = \sqrt{x}$.

(c) For $h(x) = (1 - 4x^2)^{1/2}$ find f(g) when $g(x) = 1 - 4x^2$, $g(x) = 2x$.

(d) For $h(x) = x^2 + 2x + 5$ find f(g) when $g(x) = \sqrt{x}$, $g(x) = x + 1$.

(e) For $h(x) = x^{2/3} - x^{4/3} + 1$ find f(g) when $g(x) = x^{2/3} - x^{4/3}$, $g(x) = x^{1/3}$.

(f) For $h(x) = \sqrt{x^{1/2} + x^2}$ find f(g) when $g(x) = x^{1/2}$, $g(x) = x^2$.

(g) For $h(x) = \frac{x^2 + x + 1}{x^2 + x + 2}$ find f(g) when $g(x) = x^2 + x$, $g(x) = 2x$.

(4) For each of the following choices of f(g) and g(x), sketch a fairly accurate graph of f(g(x)):

(a) $f(g) = -g + 2$ and $g(x) = \sin(x) + 1$

(b) $f(g) = \sqrt{g}$ and $g(x) = 3\sin(x)$

(c) $f(g) = |g|$ and $g(x) = \cos(x)$

(5) Compute the following derivatives:

(a) $\frac{d}{dt}(t) =$

(b) $\frac{d}{dx}(x + 2/(x - 1)^{2/3}) =$

(c) $\frac{d}{dx}(x^2 + x + 1)^5 =$

(d) $\frac{d}{dx}\left(\left(\frac{2}{x^2 + 1}\right)^8\right) =$

(e) If $g(t) = \sqrt{3t^2 - 4}$ then $g'(t) =$

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(f) Find $\frac{df}{dx}$ if $f(x) = \frac{1}{(3x^2 - 5x^2)^2}$.

(g) $\frac{d}{dy}(y^4 - (y + 1)^3)$

(h) If $C(x) = (A(x) - 3B(x))^4$ find $C'(-2)$ if $A(-2) = B(-2) = 1$, $A'(-2) = 0$, and $B'(-2) = -2$.

(i) Find the equation of the line tangent to the curve $y = \sqrt{x^2 - 1}$ at $x = 2$.

(j) Find the equation of the line normal to the curve $y = x^4 + 2$ at $x = -1$.

(k) For $f(x) = 1/\sqrt{x + 1}$ find $f'(x)$ and $f''(x)$.

(l) For $g(t) = t^3 + 2t^2 + t + 4$ find the "third derivative" $\frac{d^3g(t)}{dt^3}$.

(m) Find $\left[\frac{d}{dx} f(x^2 + x + 1)\right]_{x=0}$ if $f'(1) = 2$.

1.21 VARIATIONS ON EXERCISE 1.16

(1) Find the compositions indicated below. In the cases indicated, specify the values for which the composition is defined.

(a) Find $h(x) = f(g(x))$ where $f(x) = (x - 1)^2 + 4x + 1$ and $g(x) = x^3 + 2x$.

(b) Find $h(x) = f(g(x))$ where $f(s) = 3s^3 + 5$ and $g(x) = (-x + 4)^3$.

(c) Find $h(x) = s(t(x))$ where $s(x) = x^4 - x^3 + 2x + 1$ and $t(x) = x^2 + 2x - 1$.

(d) Find $P(q(y))$ where $P(q) = (q^2 - 1)/(q^2 + 1)$ and $q(y) = \sqrt{y}$.

(e) Find $h(x) = f(g(x))$ where $f(g) = \frac{g}{g - 1}$ and $g(x) = x^{1/3}$. For what values of x is h(x) defined?

(f) Find $h(k(t))$ and $k(h(t))$ where $h(t) = t^2 - 1$ and $k(t) = \sqrt{t + 1}$. For what values of t are these functions defined?

(g) Find $f(g(x))$ and $g(f(x))$ if $f(x) = x^2 + x$ and $g(x) = x - 1$.

(2) Find functions f(g) and g(x) such that $h(x) = f(g(x))$ in each of the following cases:

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(a) $h(x) = (x^2 + x - 1)^3 + \sqrt{x^2 + x - 1} + 4$

(b) $h(x) = (x^3 - 1)^2 + x^3 - 2$

(c) $h(x) = ((x - 2)^2 - 1)/((x - 2)^2 + 1)$

(d) $h(x) = (x^2 + 8x + 3)/((x - 4)^2 + 1)$

(3) In each of the following cases a function h(x) is given. For each of the specified functions g(x), find f(g) so that $f(g(x)) = h(x)$.

(a) For $h(x) = (x + 3)^2 - x + 2$, find f(g) when $g(x) = x + 3$.

(b) For $h(x) = x^2 + x^3$, find f(g) when $g(x) = x^2$ and $g(x) = x^{1/2}$.

(c) For $h(x) = \sqrt{2x^2 - 3}$, find f(g) when $g(x) = 2x^2 - 3$ and $g(x) = 2x^2$.

(d) For $h(x) = x/(x^{1/3} + x^2)$, find f(g) when $g(x) = x^{1/3}$ and $g(x) = x^{1/3} + 1$.

(e) For $h(x) = (3 + x + 2\sqrt{x})^{5/2}$, find f(g) when $g(x) = \sqrt{x}$, and $g(x) = 1 + x + 2\sqrt{x}$.

(f) For $h(x) = (x^{1/2} - x^{-1/2})/(x^{1/2} + x^{-1/2})$, find f(g) when $g(x) = x^{1/2}$ and $g(x) = x^{-1/2}$

(4) For each of the following choices of f(g) and g(x), sketch the graph of $h(x) = f(g(x))$. Be fairly accurate.

(a) $f(g) = g^2 + 1$ and $g(x) = |x|$

(b) $f(g) = |g|$ and $g(x) = \cos(x)$

(c) $f(g) = \sqrt{g}$ and $g(x) = |x|$

(5) Compute the following derivatives:

(a) $\frac{d}{dx}(3) =$

(b) $\frac{d}{dx}(x^4 + 3x^3 + 2x^2 + x - 1) =$

(c) $\frac{d}{dt}(t^4 + 1)^8 =$

(d) $\frac{d}{ds}\left(\frac{2}{s^2 + 1}\right)^3 =$

(e) Find $g'(x)$ if $g(x) = \sqrt{\sqrt{x} + 1}$.

(f) $\frac{d^2}{dx^2}(x - x^{-1}) =$

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(g) Find $g'(x)$ and $g''(x)$ if $g(x) = 1/x^2$.

(h) If $f(x) = (g(x))^2 + 1$ find $f'(2)$ if $g(2) = 1$, $g'(2) = -1$.

(i) Find the equation of the line tangent to $y = \sqrt{x}$ at $x = 4$.

(j) Find the line normal to the curve $y = 2x^2 - 3x + 1$ at $x = -1$.

(k) Find $g^{(3)}(0)$ if $g(x) = x^4 - x^3 + x^2 - x + 1$.

1.22 VARIATIONS ON EXERCISE 1.16

(1) Find the compositions indicated below. In the cases indicated, specify the values for which the composition is defined.

(a) Find $h(x) = f(g(x))$ where $f(x) = 2x/\sqrt{1 - x^2}$ and $g(x) = 1 + x^2$.

(b) Find $g(f(x))$ where $g(t) = 3 + \cos^2 t$ and $f(x) = x^2 - 1$.

(c) Find $r(s(t))$ where $r(y) = 1/y + 2/y^2 - 3/y^4$ and $s(t) = 2 + 1/t$.

(d) Find $a(b(x))$ where $a(x) = \sin^2 x + 1$ and $b(x) = 1 + \cos x$.

(e) Find $h(t) = f(g(t))$ where $f(t) = \sqrt{t}$ and $g(t) = -t$. For what values of t is $h(t)$ defined?

(f) Find $f(g(x))$ and $g(f(x))$ where $f(x) = \sqrt{x}$ and $g(x) = \sin x$. For what values of x between $-\pi$ and π are these two functions defined?

(g) Find $f(g(x))$ and $g(f(x))$ where $f(x) = -x^2$ and $g(x) = |x|$.

(2) Find functions $f(g)$ and $g(x)$ such that $h(x) = f(g(x))$ in each of the following cases:

(a) $h(x) = \sin^2(x^2 + 4x + 1)$

(b) $h(x) = \sqrt{2x^2 + 1} - 4$

(c) $h(x) = (2x - 3)^2 + 5x - 3$

(d) $h(x) = 2x^2 - 18x + 4 - (x - 3)^{2/3}$

(3) In each of the following cases a function $h(x)$ is given. For each of the specified functions $g(x)$, find $f(g)$ so that $h(x) = f(g(x))$.

(a) For $h(x) = \sin^3(x^2 + 1)$ find $f(g)$ when $g(x) = x^2 + 1$ and $g(x) = \sin(x^2 + 1)$.

(b) For $h(x) = x^2 - x^{-2}$ find $f(g)$ when $g(x) = x^2$ and $g(x) = x^{-1/3}$.

(c) For $h(x) = \sqrt{(x - 1)^2 + 3}$ find $f(g)$ when $g(x) = (x - 1)$ and $g(x) = (x - 1)^2$.

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(d) For $h(x) = \cos(\sqrt{x+1})$ find $f(g)$ when $g(x) = x + 1$ and $g(x) = \sqrt{x+1}$.

(e) For $h(x) = \sqrt{(x^2 - 4)^3}$ find $f(g)$ when $g(x) = x^2$ and $g(x) = x^2 - 4$.

(f) For $h(x) = \dfrac{1}{3x^2 + 4x + 1}$ find $f(g)$ when $g(x) = 3x^2 + 4x + 1$ and $g(x) = 3x^2 + 4x - 1$.

(4) For each of the following choices of $f(g)$ and $g(x)$, sketch a reasonably accurate graph of $f(g(x))$:

(a) $f(g) = \sin(g)$ and $g(x) = 2x + \pi$

(b) $f(g) = g^2 + 1$ and $g(x) = \cos x$

(c) $f(g) = 1 - 3g$ and $g(x) = |x|$

(5) Compute the following derivatives:

(a) $\dfrac{d}{dx}((1 + x^2)^{-1}) =$

(b) $\dfrac{d}{dz}(2z^2 - 4/z)^2 =$

(c) If $g(x) = \sqrt{2x^3 + 1}$ then $g'(1) =$

(d) $\dfrac{d}{dd}(2) =$

(e) $\dfrac{d}{dx}\left(\dfrac{2}{1 + |x|}\right) =$

(f) Find $f'(x)$, $f''(x)$, and $f(x) = (x - 1)^{10}$.

(g) $\dfrac{d^2}{dt^2}(1 + t^{-1})^{-1} =$

(h) Find the equation of the line tangent to $y = 1/x$ at $x = -1$.

(i) Find the equation of the line normal to $y = \sqrt{x - 1}$ at $x = 2$.

(j) If $f(x) = \sqrt{(g(x))^2 + 1}$ find $f'(0)$ if $g(0) = 1$, $g'(0) = 2$.

(k) Find $\dfrac{d^3 f(x)}{dx^3}$ if $f(x) = x^4 - 2x^2$.

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