Top-down Calculus

Appendix 1

Math Tables

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Preface

Appendix 1 of *Top-down Calculus* consists of math tables for derivatives, series, products and integrals. Also included are tables of identities for trigonometric and hyperbolic functions. Tables of integrals and integrators/calculators are available online. Occasionally, you might wish to print out parts of Appendix 1 to take a break from looking at a computer screen. We have kept the original page numbering from *Top-down Calculus* for easy cross reference.

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Appendix 1

MATH TABLES

DERIVATIVES

$$\frac{d(au)}{dx} = \frac{a\,du}{dx}.$$

$$\frac{d(u+v)}{dx} = \frac{du}{dx} + \frac{dv}{dx}.$$

$$\frac{d(uv)}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}.$$

$$\frac{d\left(\frac{u}{v}\right)}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}.$$

$$\frac{df(u)}{dx} = \frac{df(u)}{du}\cdot\frac{du}{dx}.$$

$$\frac{d^2f(u)}{dx^2} = \frac{df}{du}\cdot\frac{d^2u}{dx^2} + \frac{d^2f}{du^2}\cdot\frac{du^2}{dx^2}.$$

$$\frac{dx^n}{dx} = nx^{n-1}.$$

$$\frac{de^x}{dx} = e^x.$$

$$\frac{da^u}{dx} = a^u\cdot\frac{du}{dx}\cdot\log_e a.$$

$$\frac{dx^x}{dx} = x^x(1 + \log_e x).$$

$$\frac{d(\log_a x)}{dx} = \frac{1}{x\cdot\log_e a} = \frac{\log_a e}{x}.$$

$$\frac{d\sin x}{dx} = \cos x.$$

$$\frac{d\cos x}{dx} = -\sin x.$$

$$\frac{d\tan x}{dx} = \sec^2 x.$$

$$\frac{d\operatorname{ctn} x}{dx} = -\csc^2 x.$$

$$\frac{d\sec x}{dx} = \tan x\cdot\sec x.$$

$$\frac{d\csc x}{dx} = -\operatorname{ctn} x\cdot\csc x.$$

$$\frac{d\sin^{-1}x}{dx} = \frac{1}{\sqrt{1-x^2}}.$$

$$\frac{d\cos^{-1}x}{dx} = \frac{-1}{\sqrt{1-x^2}}.$$

$$\frac{d\tan^{-1}x}{dx} = \frac{1}{1+x^2}.$$

$$\frac{d\,\mathrm{ctn}^{-1}x}{dx} = -\frac{1}{1+x^2}.$$

$$\frac{d\sec^{-1}x}{dx} = \frac{1}{x\sqrt{x^2-1}}.$$

$$\frac{d\csc^{-1}x}{dx} = -\frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d\sinh x}{dx} = \cosh x.$$

$$\frac{d\cosh x}{dx} = \sinh x.$$

$$\frac{d\tanh x}{dx} = \mathrm{sech}^2 x.$$

$$\frac{d\,\mathrm{ctnh}\,x}{dx} = -\,\mathrm{csch}^2 x.$$

$$\frac{d\,\mathrm{sech}\,x}{dx} = -\,\mathrm{sech}\,x\cdot\tanh x.$$

$$\frac{d\,\mathrm{csch}\,x}{dx} = -\,\mathrm{csch}\,x\cdot\mathrm{ctnh}\,x.$$

$$\frac{d\sinh^{-1}x}{dx} = \frac{1}{\sqrt{x^2+1}}.$$

$$\frac{d\cosh^{-1}x}{dx} = \frac{1}{\sqrt{x^2-1}}.$$

$$\frac{d\tanh^{-1}x}{dx} = \frac{1}{1-x^2}.$$

$$\frac{d\,\mathrm{ctnh}^{-1}x}{dx} = \frac{1}{1-x^2}.$$

$$\frac{d\,\mathrm{sech}^{-1}x}{dx} = \frac{-1}{x\sqrt{1-x^2}}.$$

$$\frac{d\,\mathrm{csch}^{-1}x}{dx} = \frac{-1}{x\sqrt{x^2+1}}.$$

$$\frac{d}{db}\int_a^b f(x)\,dx = f(b).$$

$$\frac{d}{da}\int_a^b f(x)\,dx = -f(a).$$

SERIES AND PRODUCTS

[The expression in brackets attached to an infinite series shows values of the variable which lie within the interval of convergence. If a series is convergent for all finite values of x, the expression $[x^2 < \infty]$ is used.]

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \cdots + \frac{n!\,a^{n-k}b^k}{(n-k)!\,k!} + \cdots. \quad [b^2 < a^2.]$$

$$(a-bx)^{-1} = \frac{1}{a}\left[1 + \frac{bx}{a} + \frac{b^2x^2}{a^2} + \frac{b^3x^3}{a^3} + \cdots\right]. \quad [b^2x^2 < a^2.]$$

$$(1\pm x)^n = 1 \pm nx + \frac{n(n-1)}{2!}x^2 \pm \frac{n(n-1)(n-2)x^3}{3!} + \cdots + \frac{(\pm 1)^k n!\,x^k}{(n-k)!\,k!} + \cdots. \quad [x^2 < 1.]$$

$$(1 \pm x)^{-n} = 1 \mp nx + \frac{n(n+1)}{2!}x^2 \mp \frac{n(n+1)(n+2)x^3}{3!} + \cdots (\mp)^k \frac{(n+k-1)!\,x^k}{(n-1)!\,k!} + \cdots. \quad [x^2 < 1.]$$

$$(1 \pm x)^{\frac{1}{2}} = 1 \pm \tfrac{1}{2}x - \frac{1\cdot 1}{2\cdot 4}x^2 \pm \frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 6}x^3 - \frac{1\cdot 1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 8}x^4 \pm \cdots. \quad [x^2 < 1.]$$

$$(1 \pm x)^{-\frac{1}{2}} = 1 \mp \tfrac{1}{2}x + \frac{1\cdot 3}{2\cdot 4}x^2 \mp \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^3 + \frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}x^4 \mp \cdots. \quad [x^2 < 1.]$$

$$(1 \pm x)^{\frac{1}{3}} = 1 \pm \tfrac{1}{3}x - \frac{1\cdot 2}{3\cdot 6}x^2 \pm \frac{1\cdot 2\cdot 5}{3\cdot 6\cdot 9}x^3 - \frac{1\cdot 2\cdot 5\cdot 8}{3\cdot 6\cdot 9\cdot 12}x^4 \pm \cdots. \quad [x^2 < 1.]$$

SERIES

$$(1 \pm x)^{-\frac{1}{3}} = 1 \mp \tfrac{1}{3}x + \frac{1\cdot 4}{3\cdot 6}x^2 \mp \frac{1\cdot 4\cdot 7}{3\cdot 6\cdot 9}x^3 + \frac{1\cdot 4\cdot 7\cdot 10}{3\cdot 6\cdot 9\cdot 12}x^4 \mp \cdots. \quad [x^2 < 1.]$$

$$(1 \pm x^2)^{\frac{1}{2}} = 1 \pm \tfrac{1}{2}x^2 - \frac{x^4}{2\cdot 4} \pm \frac{1\cdot 3x^6}{2\cdot 4\cdot 6} - \frac{1\cdot 3\cdot 5x^8}{2\cdot 4\cdot 6\cdot 8} \pm \cdots. \quad [x^2 < 1.]$$

$$(1 \pm x^2)^{-\frac{1}{2}} = 1 \mp \tfrac{1}{2}x^2 + \frac{1\cdot 3}{2\cdot 4}x^4 \mp \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^6 + \cdots. \quad [x^2 < 1.]$$

$$(1 \pm x)^{-1} = 1 \mp x + x^2 \mp x^3 + x^4 \mp x^5 + \cdots. \quad [x^2 < 1.]$$

$$(1 \pm x)^{\frac{3}{2}} = 1 \pm \tfrac{3}{2}x + \frac{3\cdot 1}{2\cdot 4}x^2 \mp \frac{3\cdot 1\cdot 1}{2\cdot 4\cdot 6}x^3 + \frac{3\cdot 1\cdot 1\cdot 3}{2\cdot 4\cdot 6\cdot 8}x^4 \mp \frac{3\cdot 1\cdot 1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 8\cdot 10}x^5 + \cdots. \quad [x^2 < 1.]$$

$$(1 \pm x)^{-\frac{3}{2}} = 1 \mp \tfrac{3}{2}x + \frac{3\cdot 5}{2\cdot 4}x^2 \mp \frac{3\cdot 5\cdot 7}{2\cdot 4\cdot 6}x^3 + \cdots. \quad [x^2 < 1.]$$

$$(1 \pm x)^{-2} = 1 \mp 2x + 3x^2 \mp 4x^3 + 5x^4 \mp 6x^5 + \cdots. \quad [x^2 < 1.]$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots. \quad [x^2 < \infty.]$$

$$a^x = 1 + x \log a + \frac{(x \log a)^2}{2!} + \frac{(x \log a)^3}{3!} + \cdots. \quad [x^2 < \infty.]$$

$$\tfrac{1}{2}(e^x + e^{-x}) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots. \quad [x^2 < \infty.]$$

$$\tfrac{1}{2}(e^x - e^{-x}) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots. \quad [x^2 < \infty.]$$

$$e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} - \cdots. \quad [x^2 < \infty.]$$

$$\log x = 2\left[\frac{x-1}{x+1} + \tfrac{1}{3}\left(\frac{x-1}{x+1}\right)^3 + \tfrac{1}{5}\left(\frac{x-1}{x+1}\right)^5 + \cdots\right]. \quad [x > 0.]$$

$$\log(1+x) = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 + \cdots. \quad [x^2 < 1.]$$

$$\log\left(\frac{1+x}{1-x}\right) = 2\left[x + \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 + \tfrac{1}{7}x^7 + \cdots\right]. \quad [x^2 < 1.]$$

$$\log\left(\frac{x+1}{x-1}\right) = 2\left[\frac{1}{x} + \tfrac{1}{3}\left(\frac{1}{x}\right)^3 + \tfrac{1}{5}\left(\frac{1}{x}\right)^5 + \cdots\right]. \quad [x^2 > 1.]$$

$$\log(x + \sqrt{1+x^2}) = x - \frac{1\,x^3}{6} + \frac{1\cdot 3\,x^5}{2\cdot 4\cdot 5} - \frac{1\cdot 3\cdot 5\,x^7}{2\cdot 4\cdot 6\cdot 7} + \cdots. \quad [x^2 < 1.]$$

Series for denary and other logarithms can be obtained from the foregoing developments by aid of the equations,

$$\log_a x = \log_e x \cdot \log_a e, \quad \log_e x = \log_a x \cdot \log_e a,$$
$$\log_e(-z) = (2n+1)\pi i + \log_e z.$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots. \quad [x^2 < \infty.]$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = 1 - \text{versin}\, x. \quad [x^2 < \infty.]$$

$$\tanh^{-1}x = x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \cdots. \qquad [x^2 < 1.]$$

$$\text{ctnh}^{-1}\,x = \frac{1}{x} + \frac{1}{3\,x^3} + \frac{1}{5\,x^5} + \cdots. \qquad [x^2 > 1.]$$

$$\text{csch}^{-1}\,x = \frac{1}{x} - \frac{1}{2\cdot 3\cdot x^3} + \frac{1\cdot 3}{2\cdot 4\cdot 5\cdot x^5} - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 7\cdot x^7} + \cdots. \qquad [x^2 > 1.]$$

$$\int_0^x e^{-x^2}\,dx = x - \tfrac{1}{3}x^3 + \frac{x^5}{5\cdot 2!} - \frac{x^7}{7\cdot 3!} + \cdots. \qquad [x^2 < \infty.]$$

$$\int_0^x \cos(x^2)\,dx = x - \frac{x^5}{5\cdot 2!} + \frac{x^9}{9\cdot 4!} - \frac{x^{13}}{13\cdot 6!} + \cdots. \qquad [x^2 < \infty.]$$

$$\int_0^1 \frac{x^{a-1}dx}{1+x^b} = \frac{1}{a} - \frac{1}{a+b} + \frac{1}{a+2\,b} - \frac{1}{a+3\,b} + \cdots.$$

$$f(x+h) = f(x) + h\cdot f'(x+\theta h).$$

$$f(x+h) = f(x) + h\cdot f'(x) + \frac{h^2}{2!}f''(x) + \cdots + \frac{h^n}{n!}\cdot f^n(x+\theta h).$$

$$f(x+h) = f(x) + h\cdot f'(x) + \frac{h^2}{2!}f''(x) + \cdots + \frac{h^n}{(n-1)!}\cdot(1-\theta)^{n-1}\cdot f^n(x+\theta h).$$

$$f(x+h,\,y+k) = f(x,\,y) + hf'_x(x+\theta h,\,y+\theta k) + kf'_y(x+\theta h,\,y+\theta k).$$

$$\sin^{-1}x = x + \frac{x^3}{6} + \frac{1\cdot 3}{2\cdot 4}\cdot\frac{x^5}{5} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{x^7}{7} + \cdots = \tfrac{1}{2}\pi - \cos^{-1}x. \qquad [x^2 < 1.]$$

$$\tan^{-1}x = x - \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 - \tfrac{1}{7}x^7 + \cdots = \tfrac{1}{2}\pi - \text{ctn}^{-1}x. \qquad [x^2 < 1.]$$

$$\tan^{-1}x = \frac{\pi}{2} - \frac{1}{x} + \frac{1}{3\,x^3} - \frac{1}{5\,x^5} + \cdots. \qquad [x^2 > 1.]$$

$$\sec^{-1}x = \frac{\pi}{2} - \frac{1}{x} - \frac{1}{6x^3} - \frac{1\cdot 3}{2\cdot 4\cdot 5\,x^5} - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 7\,x^7} - \cdots = \tfrac{1}{2}\pi - \csc^{-1}x. \qquad [x^2 > 1.]$$

$$e^{\sin x} = 1 + x + \frac{x^2}{2!} - \frac{3x^4}{4!} - \frac{8x^5}{5!} - \frac{3x^6}{6!} + \frac{56x^7}{7!} + \cdots. \quad [x^2 < \infty.]$$

$$e^{\cos x} = e\left(1 - \frac{x^2}{2!} + \frac{4x^4}{4!} - \frac{31x^6}{6!} + \cdots\right). \quad [x^2 < \infty.]$$

$$e^{\tan x} = 1 + x + \frac{x^2}{2!} + \frac{3x^3}{3!} + \frac{9x^4}{4!} + \frac{37x^5}{5!} + \cdots. \quad [x^2 < \tfrac{1}{4}\pi^2.]$$

$$e^{\sin^{-1} x} = 1 + x + \frac{x^2}{2!} + \frac{2x^3}{3!} + \frac{5x^4}{4!} + \cdots. \quad [x^2 < 1.]$$

$$e^{\tan^{-1} x} = 1 + x + \frac{x^2}{2} - \frac{x^3}{6} - \frac{7x^4}{24} - \cdots. \quad [x^2 < 1.]$$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots. \quad [x^2 < \infty.]$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \frac{x^8}{8!} + \cdots. \quad [x^2 < \infty.]$$

A series of numbers, B_1, B_2, $B_3 \cdots$, of odd and even orders, which appear in the developments of many functions, may be computed by means of the equations,

$$B_{2n} - \frac{2n(2n-1)}{2!} B_{2n-2} + \frac{2n(2n-1)(2n-2)(2n-3)}{4!} B_{2n-4} - \cdots (-1)^n = 0.$$

$$\frac{2^{2n}(2^{2n}-1)}{2n} B_{2n-1} = (2n-1) B_{2n-2} - \frac{(2n-1)(2n-2)(2n-3)}{3!} B_{2n-4} + \cdots (-1)^{n-1} = 0.$$

$$\frac{x}{e^x - 1} = 1 - \frac{x}{2} + \frac{B_1 x^2}{2!} - \frac{B_3 x^4}{4!} + \frac{B_5 x^6}{6!} - \frac{B_7 x^8}{8!} + \cdots. \quad [x < 2\pi.]$$

$$\log x = (x-1) - \tfrac{1}{2}(x-1)^2 + \tfrac{1}{3}(x-1)^3 - \cdots. \quad [2 > x > 0.]$$

$$\log x = \frac{x-1}{x} + \tfrac{1}{2}\left(\frac{x-1}{x}\right)^2 + \tfrac{1}{3}\left(\frac{x-1}{x}\right)^3 + \cdots. \quad [x > \tfrac{1}{2}.]$$

$$\text{ctnh}\, x = \frac{1}{x}(1 + \Sigma[(-1)^{n-1} 2^{2n} B_{2n-1} x^{2n}/(2n)!]). \qquad [x^2 < \pi^2.]$$

$$\text{sech}\, x = 1 + \Sigma[(-1)^n B_{2n} x^{2n}/(2n)!]. \qquad [x^2 < \tfrac{1}{4}\pi^2.]$$

$$\text{csch}\, x = \frac{1}{x} - (2-1)\, 2\, B_1 \frac{x}{2!} + (2^3 - 1)\, 2\, B_3 \frac{x^3}{4!} - \cdots$$

$$= \frac{1}{x}(1 + 2\,\Sigma[(-1)^n (2^{2n-1} - 1) B_{2n-1} x^{2n}/(2n)!]). \qquad [x^2 < \pi^2.]$$

$$\sinh^{-1} x = x - \tfrac{1}{6} x^3 + \frac{1 \cdot 3 \cdot x^5}{2 \cdot 4 \cdot 5} - \frac{1 \cdot 3 \cdot 5 \cdot x^7}{2 \cdot 4 \cdot 6 \cdot 7} + \cdots. \quad [x^2 < 1.]$$

$$\log \sin x = \log x - \tfrac{1}{6} x^2 - \tfrac{1}{180} x^4 - \tfrac{1}{2835} x^6 - \cdots - \frac{2^{2n-1} B_{2n-1} x^{2n}}{n(2n)!} - \cdots. \qquad [x^2 < \pi^2.]$$

$$\log \cos x = -\tfrac{1}{2} x^2 - \tfrac{1}{12} x^4 - \tfrac{1}{45} x^6 - \tfrac{17}{2520} x^8 - \cdots - \frac{2^{2n-1}(2^{2n} - 1) B_{2n-1} x^{2n}}{n(2n)!} - \cdots. \qquad [x^2 < \tfrac{1}{4}\pi^2.]$$

$$\log \tan x = \log x + \tfrac{1}{3} x^2 + \tfrac{7}{90} x^4 + \tfrac{62}{2835} x^6 + \cdots + \frac{(2^{2n-1} - 1)\, 2^{2n} B_{2n-1} x^{2n}}{n(2n)!} + \cdots. \qquad [x^2 < \tfrac{1}{4}\pi^2.]$$

Whence $B_1 = \frac{1}{6}$, $B_2 = 1$, $B_3 = \frac{1}{30}$, $B_4 = 5$, $B_5 = \frac{1}{42}$, $B_6 = 61$, $B_7 = \frac{1}{30}$, $B_8 = 1385$, $B_9 = \frac{5}{66}$, $B_{10} = 50521$, $B_{11} = \frac{691}{2730}$, $B_{12} = 2702765$, $B_{13} = \frac{7}{6}$, etc. The B's of odd orders are called Bernoulli's Numbers; those of even orders, Euler's Numbers. What are here denoted by B_{2n-1} and B_{2n} are sometimes represented by B_n and E_n, respectively,

$$\frac{B_{2n-1}}{(2n)!} = \frac{2}{(2^{2n} - 1)\pi^{2n}}\left[1 + \frac{1}{3^{2n}} + \frac{1}{5^{2n}} + \frac{1}{7^{2n}} + \cdots\right],$$

$$\frac{B_{2n}}{(2n)!} = \frac{2^{2n+2}}{\pi^{2n+1}}\left[1 - \frac{1}{3^{2n+1}} + \frac{1}{5^{2n+1}} - \frac{1}{7^{2n+1}} + \cdots\right].$$

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \frac{62x^9}{2835}$$

$$+ \cdots + \frac{2^{2n}(2^{2n}-1)B_{2n-1}x^{2n-1}}{(2n)!} + \cdots. \qquad [x^2 < \tfrac{1}{4}\pi^2.]$$

$$\operatorname{ctn} x = \frac{1}{x} - \frac{x}{3} - \frac{x^3}{45} - \frac{2x^5}{945} - \frac{x^7}{4725}$$

$$- \cdots - \frac{B_{2n-1}(2x)^{2n}}{x(2n)!} - \cdots. \qquad [x^2 < \pi^2.]$$

$$\sec x = 1 + \frac{x^2}{2!} + \frac{5x^4}{4!} + \frac{61x^6}{6!} + \cdots + \frac{B_{2n}x^{2n}}{(2n)!} + \cdots. \left[x^2 < \frac{\pi^2}{4}.\right]$$

$$\csc x = \frac{1}{x} + \frac{x}{3!} + \frac{7x^3}{3\cdot 5!} + \frac{31x^5}{3\cdot 7!}$$

$$+ \cdots + \frac{2(2^{2n+1}-1)}{(2n+2)!}B_{2n+1}x^{2n+1} + \cdots. \qquad [x^2 < \pi^2.]$$

$$\tanh x = (2^2-1)2^2 B_1 \frac{x}{2!} - (2^4-1)2^4 B_3 \frac{x^3}{4!} + \cdots$$

$$= \Sigma[(-1)^{n-1}2^{2n}(2^{2n}-1)B_{2n-1}x^{2n-1}/(2n)!]. \qquad [x^2 < \tfrac{1}{4}\pi^2.]$$

$$\log \sin \tfrac{1}{2}x = -\log 2 - \cos x - \tfrac{1}{2}\cos 2x - \tfrac{1}{3}\cos 3x - \cdots. \qquad [0 < x < \tfrac{1}{2}\pi.]$$

$$\log \cos \tfrac{1}{2}x = -\log 2 + \cos x - \tfrac{1}{2}\cos 2x + \tfrac{1}{3}\cos 3x - \cdots. \qquad [0 < x < \tfrac{1}{2}\pi.]$$

$$f(x) = \tfrac{1}{2}b_0 + b_1\cos\frac{\pi x}{c} + b_2\cos\frac{2\pi x}{c} + \cdots$$

$$+ a_1\sin\frac{\pi x}{c} + a_2\sin\frac{2\pi x}{c} + \cdots, \quad [-c < x < c.]$$

where $b_m = \frac{1}{c}\int_{-c}^{+c} f(a)\cos\frac{m\pi a}{c}\,da,$

$$a_m = \frac{1}{c}\int_{-c}^{+c} f(a)\sin\frac{m\pi a}{c}\,da.$$

TABLE OF INTEGRALS

Fundamental Forms

$$\int a\,dx = ax.$$

$$\int af(x)\,dx = a\int f(x)\,dx.$$

$$\int \frac{dx}{x} = \log x. \quad [\log x = \log(-x) + (2k+1)\pi i.]$$

$$\int x^m dx = \frac{x^{m+1}}{m+1}, \text{ when } m \text{ is different from } -1.$$

$$\int e^x dx = e^x.$$

$$\int a^x \log a\,dx = a^x.$$

$$\int \frac{dx}{1+x^2} = \tan^{-1}x, \text{ or } -\operatorname{ctn}^{-1}x.$$

$$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1}x, \text{ or } -\cos^{-1}x$$

$$\int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1}x, \text{ or } -\csc^{-1}x.$$

$$\int \frac{dx}{\sqrt{2x-x^2}} = \operatorname{versin}^{-1}x, \text{ or } -\operatorname{coversin}^{-1}x.$$

$$\int \cos x\,dx = \sin x, \text{ or } -\operatorname{coversin} x.$$

$$\int \sin x\,dx = -\cos x, \text{ or } \operatorname{versin} x.$$

$$\int \operatorname{ctn} x\,dx = \log \sin x.$$

$$\int \tan x\,dx = -\log \cos x.$$

$$\int \tan x \sec x\,dx = \sec x.$$

$$\int \sec^2 x\, dx = \tan x.$$

$$\int \csc^2 x\, dx = -\operatorname{ctn} x.$$

In the following formulas, u, v, w, and y represent any functions of x:

$$\int (u + v + w + \text{etc.})\, dx = \int u\, dx + \int v\, dx + \int w\, dx + \text{etc.}$$

$$\int u\, dv = uv - \int v\, du.$$

$$\int u \frac{dv}{dx}\, dx = uv - \int v \frac{du}{dx}\, dx.$$

$$\int f(y)\, dx = \int \frac{f(y)\, dy}{\frac{dy}{dx}}$$

Rational Algebraic Functions

Expressions Involving $(a + bx)$.

The substitution of y or z for x, where $y \equiv a + bx$, $z \equiv (a + bx)/x$, gives

$$\int (a + bx)^m\, dx = \frac{1}{b}\int y^m\, dy.$$

$$\int x\,(a + bx)^m\, dx = \frac{1}{b^2}\int y^m\,(y - a)\, dy.$$

$$\int x^n\,(a + bx)^m\, dx = \frac{1}{b^{n+1}}\int y^m\,(y - a)^n\, dy.$$

$$\int \frac{x^n\, dx}{(a + bx)^m} = \frac{1}{b^{n+1}}\int \frac{(y - a)^n\, dy}{y^m}.$$

$$\int \frac{dx}{x^n\,(a + bx)^m} = -\frac{1}{a^{m+n-1}}\int \frac{(z - b)^{m+n-2}\, dz}{z^m}.$$

Whence

$$\int \frac{dx}{a + bx} = \frac{1}{b}\log(a + bx).$$

$$\int \frac{dx}{(a+bx)^2} = -\frac{1}{b(a+bx)}.$$

$$\int \frac{dx}{(a+bx)^3} = -\frac{1}{2b(a+bx)^2}.$$

$$\int \frac{x\,dx}{a+bx} = \frac{1}{b^2}[a+bx-a\log(a+bx)].$$

$$\int \frac{x\,dx}{(a+bx)^2} = \frac{1}{b^2}\left[\log(a+bx)+\frac{a}{a+bx}\right].$$

Expressions Involving $(a+bx^n)$.

$$\int \frac{dx}{c^2+x^2} = \frac{1}{c}\tan^{-1}\frac{x}{c} = \frac{1}{c}\sin^{-1}\frac{x}{\sqrt{x^2+c^2}}.$$

$$\int \frac{dx}{c^2-x^2} = \frac{1}{2c}\log\frac{c+x}{c-x}, \quad \int \frac{dx}{x^2-c^2} = \frac{1}{2c}\log\frac{x-c}{x+c}.^*$$

$$\int \frac{dx}{a+bx^2} = \frac{1}{\sqrt{ab}}\tan^{-1}\left(x\sqrt{\frac{b}{a}}\right), \text{or } \frac{1}{\sqrt{-ab}}\cdot\tanh^{-1}\left(x\sqrt{\frac{-b}{a}}\right).$$

$$\int \frac{dx}{a+bx^2} = \frac{1}{2\sqrt{-ab}}\log\frac{\sqrt{a}+x\sqrt{-b}}{\sqrt{a}-x\sqrt{-b}}, \text{ if } a>0,\ b<0.$$

$$\int \frac{dx}{(a+bx^2)^2} = \frac{x}{2a(a+bx^2)} + \frac{1}{2a}\int\frac{dx}{a+bx^2}.$$

$$\int \frac{dx}{(a+bx^2)^{m+1}} = \frac{1}{2ma}\,\frac{x}{(a+bx^2)^m} + \frac{2m-1}{2ma}\int\frac{dx}{(a+bx^2)^m}.$$

$$\int \frac{x\,dx}{a+bx^2} = \frac{1}{2b}\log\left(x^2+\frac{a}{b}\right).$$

$$\int \frac{x\,dx}{(a+bx^2)^{m+1}} = \frac{1}{2}\int\frac{dz}{(a+bz)^{m+1}}, \text{ where } z=x^2.$$

$$\int \frac{dx}{x(a+bx^2)} = \frac{1}{2a}\log\frac{x^2}{a+bx^2}.$$

$$\int \frac{x^2\,dx}{a+bx^2} = \frac{x}{b} - \frac{a}{b}\int\frac{dx}{a+bx^2}.$$

$$\int \frac{dx}{x^2(a+bx^2)} = -\frac{1}{ax} - \frac{b}{a}\int\frac{dx}{a+bx^2}.$$

$$\int \frac{x^2 dx}{(a+bx^2)^{m+1}} = \frac{-x}{2mb(a+bx^2)^m} + \frac{1}{2mb}\int \frac{dx}{(a+bx^2)^m}.$$

$$\int \frac{dx}{x^2(a+bx^2)^{m+1}} = \frac{1}{a}\int \frac{dx}{x^2(a+bx^2)^m} - \frac{b}{a}\int \frac{dx}{(a+bx^2)^{m+1}}.$$

$$^*\int \frac{dx}{c^2-x^2} = \frac{1}{c}\tanh^{-1}\left(\frac{x}{c}\right); \int \frac{dx}{x^2-c^2} = -\frac{1}{c}\text{ctnh}^{-1}\left(\frac{x}{c}\right).$$

Expressions Involving $(a + bx + cx^2)$.

Let $X = a + bx + cx^2$ and $q = 4ac - b^2$, then

$$\int \frac{dx}{X} = \frac{2}{\sqrt{q}}\tan^{-1}\frac{2cx+b}{\sqrt{q}}, \text{ or } -\frac{2}{\sqrt{-q}}\cdot\tanh^{-1}\frac{2cx+b}{\sqrt{-q}}.$$

$$\int \frac{dx}{X} = \frac{1}{\sqrt{-q}}\log\frac{2cx+b-\sqrt{-q}}{2cx+b+\sqrt{-q}}, \text{ when } q<0.$$

$$\int \frac{dx}{X^2} = \frac{2cx+b}{qX} + \frac{2c}{q}\int \frac{dx}{X}.$$

$$\int \frac{dx}{X^3} = \frac{2cx+b}{q}\left(\frac{1}{2X^2} + \frac{3c}{qX}\right) + \frac{6c^2}{q^2}\int \frac{dx}{X}.$$

$$\int \frac{dx}{X^{n+1}} = \frac{2cx+b}{nqX^n} + \frac{2(2n-1)c}{qn}\int \frac{dx}{X^n}.$$

$$\int \frac{x\,dx}{X} = \frac{1}{2c}\log X - \frac{b}{2c}\int \frac{dx}{X}.$$

$$\int \frac{x\,dx}{X^2} = -\frac{bx+2a}{qX} - \frac{b}{q}\int \frac{dx}{X}.$$

$$\int \frac{x\,dx}{X^{n+1}} = -\frac{2a+bx}{nqX^n} - \frac{b(2n-1)}{nq}\int \frac{dx}{X^n}.$$

$$\int \frac{x^2}{X}dx = \frac{x}{c} - \frac{b}{2c^2}\log X + \frac{b^2-2ac}{2c^2}\int \frac{dx}{X}.$$

$$\int \frac{x^2}{X^2}dx = \frac{(b^2-2ac)x+ab}{cqX} + \frac{2a}{q}\int \frac{dx}{X}.$$

$$\int \frac{x^m dx}{X^{n+1}} = -\frac{x^{m-1}}{(2n-m+1)cX^n} - \frac{n-m+1}{2n-m+1}\cdot\frac{b}{c}\int \frac{x^{m-1}dx}{X^{n+1}} + \frac{m-1}{2n-m+1}\cdot\frac{a}{c}\int \frac{x^{m-2}dx}{X^{n+1}}.$$

Irrational Algebraic Functions

Expressions Involving $\sqrt{a+bx}$.

The substitution of a new variable of integration, $y=\sqrt{a+bx}$, gives

$$\int \sqrt{a+bx}\,dx = \frac{2}{3b}\sqrt{(a+bx)^3}.$$

$$\int x\sqrt{a+bx}\,dx = -\frac{2(2a-3bx)\sqrt{(a+bx)^3}}{15b^2}.$$

$$\int x^2\sqrt{a+bx}\,dx = \frac{2(8a^2-12abx+15b^2x^2)\sqrt{(a+bx)^3}}{105b^3}$$

$$\int \frac{\sqrt{a+bx}}{x}\,dx = 2\sqrt{a+bx} + a\int\frac{dx}{x\sqrt{a+bx}}.$$

$$\int \frac{dx}{\sqrt{a+bx}} = \frac{2\sqrt{a+bx}}{b}.$$

$$\int \frac{x\,dx}{\sqrt{a+bx}} = -\frac{2(2a-bx)}{3b^2}\sqrt{a+bx}.$$

$$\int \frac{x^2\,dx}{\sqrt{a+bx}} = \frac{2(8a^2-4abx+3b^2x^2)}{15b^3}\sqrt{a+bx}.$$

$$\int \frac{dx}{x\sqrt{a+bx}} = \frac{1}{\sqrt{a}}\log\left(\frac{\sqrt{a+bx}-\sqrt{a}}{\sqrt{a+bx}+\sqrt{a}}\right),\ \text{for } a>0.$$

$$\int \frac{dx}{x\sqrt{a+bx}} = \frac{2}{\sqrt{-a}}\tan^{-1}\sqrt{\frac{a+bx}{-a}},\ \text{or}\ \frac{-2}{\sqrt{a}}\cdot\tanh^{-1}\sqrt{\frac{a+bx}{a}}.$$

$$\int \frac{x^m\,dx}{\sqrt{a+bx}} = \frac{2x^m\sqrt{a+bx}}{(2m+1)b} - \frac{2ma}{(2m+1)b}\int\frac{x^{m-1}\,dx}{\sqrt{a+bx}}.$$

$$\int \frac{dx}{x^n\sqrt{a+bx}} = -\frac{\sqrt{a+bx}}{(n-1)ax^{n-1}} - \frac{(2n-3)b}{(2n-2)a}\int\frac{dx}{x^{n-1}\sqrt{a+bx}}.$$

Expressions Involving $\sqrt{x^2 \pm a^2}$ and $\sqrt{a^2 - x^2}$.

$$\int \sqrt{x^2 \pm a^2}\,dx = \tfrac{1}{2}\left[x\sqrt{x^2 \pm a^2} \pm a^2 \log\left(x + \sqrt{x^2 \pm a^2}\right)\right].^*$$

$$\int \sqrt{a^2 - x^2}\,dx = \tfrac{1}{2}\left(x\sqrt{a^2 - x^2} + a^2 \sin^{-1}\frac{x}{a}\right).$$

$$\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \log\left(x + \sqrt{x^2 \pm a^2}\right).^*$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a}, \text{ or } -\cos^{-1}\frac{x}{a}.$$

$$\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a}\cos^{-1}\frac{a}{x}, \text{ or } \frac{1}{a}\sec^{-1}\frac{x}{a}.$$

$$\int \frac{dx}{x\sqrt{a^2 \pm x^2}} = -\frac{1}{a}\log\left(\frac{a + \sqrt{a^2 \pm x^2}}{x}\right)$$

$$\int \frac{\sqrt{a^2 \pm x^2}}{x}\,dx = \sqrt{a^2 \pm x^2} - a\log\frac{a + \sqrt{a^2 \pm x^2}}{x}.^*$$

$$\int \frac{\sqrt{x^2 - a^2}}{x}\,dx = \sqrt{x^2 - a^2} - a\cos^{-1}\frac{a}{x}.$$

$$\int \frac{x\,dx}{\sqrt{a^2 \pm x^2}} = \pm\sqrt{a^2 \pm x^2}.$$

$$\int \frac{x\,dx}{\sqrt{x^2 - a^2}} = \sqrt{x^2 - a^2}.$$

$$\int x\sqrt{x^2 \pm a^2}\,dx = \tfrac{1}{3}\sqrt{(x^2 \pm a^2)^3}.$$

$$\int x\sqrt{a^2 - x^2}\,dx = -\tfrac{1}{3}\sqrt{(a^2 - x^2)^3}.$$

* $\log\left(\frac{x + \sqrt{x^2 + a^2}}{a}\right) = \sinh^{-1}\left(\frac{x}{a}\right)$; $\log\left(\frac{x + \sqrt{x^2 - a^2}}{a}\right) = \cosh^{-1}\left(\frac{x}{a}\right)$;
$\log\left(\frac{a + \sqrt{a^2 - x^2}}{x}\right) = \operatorname{sech}^{-1}\left(\frac{x}{a}\right)$; $\log\left(\frac{a + \sqrt{a^2 + x^2}}{x}\right) = \operatorname{csch}^{-1}\left(\frac{x}{a}\right)$.

$$\int \sqrt{(x^2 \pm a^2)^3}\,dx$$

$$= \tfrac{1}{4}\left[x\sqrt{(x^2 \pm a^2)^3} \pm \frac{3a^2x}{2}\sqrt{x^2 \pm a^2} + \frac{3a^4}{2}\log\left(x + \sqrt{x^2 \pm a^2}\right)\right].^*$$

$$\int \sqrt{(a^2 - x^2)^3}\,dx$$
$$= \tfrac{1}{4}\left[x\sqrt{(a^2 - x^2)^3} + \frac{3a^2x}{2}\sqrt{a^2 - x^2} + \frac{3a^4}{2}\sin^{-1}\frac{x}{a}\right].$$

$$\int \frac{dx}{\sqrt{(x^2 \pm a^2)^3}} = \frac{\pm x}{a^2\sqrt{x^2 \pm a^2}}.$$

$$\int \frac{dx}{\sqrt{(a^2 - x^2)^3}} = \frac{x}{a^2\sqrt{a^2 - x^2}}.$$

$$\int \frac{x\,dx}{\sqrt{(x^2 \pm a^2)^3}} = \frac{-1}{\sqrt{x^2 \pm a^2}}.$$

$$\int \frac{x\,dx}{\sqrt{(a^2 - x^2)^3}} = \frac{1}{\sqrt{a^2 - x^2}}.$$

$$\int x\sqrt{(x^2 \pm a^2)^3}\,dx = \tfrac{1}{5}\sqrt{(x^2 \pm a^2)^5}.$$

$$\int x\sqrt{(a^2 - x^2)^3}\,dx = -\tfrac{1}{5}\sqrt{(a^2 - x^2)^5}.$$

$$\int x^2\sqrt{x^2 \pm a^2}\,dx$$
$$= \frac{x}{4}\sqrt{(x^2 \pm a^2)^3} \mp \frac{a^2}{8}x\sqrt{x^2 \pm a^2} - \frac{a^4}{8}\log\left(x + \sqrt{x^2 \pm a^2}\right).^*$$

$$\int x^2\sqrt{a^2 - x^2}\,dx$$
$$= -\frac{x}{4}\sqrt{(a^2 - x^2)^3} + \frac{a^2}{8}\left(x\sqrt{a^2 - x^2} + a^2\sin^{-1}\frac{x}{a}\right).$$

* $\log z = \sinh^{-1}\left(\frac{z^2-1}{2z}\right) = \cosh^{-1}\left(\frac{z^2+1}{2z}\right)$; $\tanh^{-1} z = -i \cdot \tan^{-1}(zi)$.

$$\int \frac{\sqrt{a^2 \pm x^2}\,dx}{x^3} = -\frac{\sqrt{a^2 \pm x^2}}{2x^2} \pm \frac{1}{2}\int \frac{dx}{x\sqrt{a^2 \pm x^2}}.$$

$$\int x^3\sqrt{a^2 \pm x^2}\,dx = \left(\pm\tfrac{1}{5}x^2 - \tfrac{2}{15}a^2\right)\sqrt{(a^2 \pm x^2)^3}.$$

$$\int \frac{dx}{x^3\sqrt{a^2 \pm x^2}} = -\frac{\sqrt{a^2 \pm x^2}}{2a^2x^2} \mp \frac{1}{2a^2}\int \frac{dx}{x\sqrt{a^2 \pm x^2}}.$$

$$\int \frac{dx}{x^3\sqrt{x^2-a^2}} = \frac{\sqrt{x^2-a^2}}{2a^2x^2} + \frac{1}{2a^3}\sec^{-1}\left(\frac{x}{a}\right).$$

$$\int \frac{x^2\,dx}{\sqrt{x^2 \pm a^2}} = \frac{x}{2}\sqrt{x^2 \pm a^2} \mp \frac{a^2}{2}\log\left(x + \sqrt{x^2 \pm a^2}\right).^*$$

$$\int \frac{x^2\,dx}{\sqrt{a^2-x^2}} = -\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}.$$

$$\int \frac{dx}{x^2\sqrt{x^2 \pm a^2}} = \mp\frac{\sqrt{x^2 \pm a^2}}{a^2x}.$$

$$\int \frac{dx}{x^2\sqrt{a^2-x^2}} = -\frac{\sqrt{a^2-x^2}}{a^2x}.$$

$$\int \frac{\sqrt{x^2 \pm a^2}\,dx}{x^2} = -\frac{\sqrt{x^2 \pm a^2}}{x} + \log\left(x + \sqrt{x^2 \pm a^2}\right).^*$$

$$\int \frac{\sqrt{a^2-x^2}}{x^2}\,dx = -\frac{\sqrt{a^2-x^2}}{x} - \sin^{-1}\frac{x}{a}.$$

$$\int \frac{x^2\,dx}{\sqrt{(x^2 \pm a^2)^3}} = \frac{-x}{\sqrt{x^2 \pm a^2}} + \log\left(x + \sqrt{x^2 \pm a^2}\right).^*$$

$$\int \frac{x^2\,dx}{\sqrt{(a^2-x^2)^3}} = \frac{x}{\sqrt{a^2-x^2}} - \sin^{-1}\frac{x}{a}.$$

Expressions Involving $\sqrt{a+bx+cx^2}$.

Let $X = a + bx + cx^2$, $q = 4ac - b^2$, and $k = \frac{4c}{q}\cdot$ In order to rationalize the function $f(x, \sqrt{a+bx+cx^2})$ we may put $\sqrt{a+bx+cx^2} = \sqrt{\pm c}\sqrt{A + Bx \pm x^2}$, according as c is positive or negative, and then substitute for x a new variable z, such that

$$z = \sqrt{A + Bx + x^2} \pm x, \text{ if } c > 0.$$

$$z = \frac{\sqrt{A + Bx - x^2} - \sqrt{A}}{x}, \text{ if } c < 0 \text{ and } \frac{a}{-c} > 0.$$

$z = \sqrt{\dfrac{x-\beta}{\alpha - x}}$, where α and β are the roots of the equation

$$A + Bx - x^2 = 0, \text{ if } c < 0 \text{ and } \frac{a}{-c} < 0.$$

By rationalization, or by the aid of reduction formulas, may be obtained the values of the following integrals:

$$\int \frac{dx}{\sqrt{X}} = \frac{1}{\sqrt{c}} \log\left(\sqrt{X} + x\sqrt{c} + \frac{b}{2\sqrt{c}}\right), \text{ if } c > 0.$$

$$\int \frac{dx}{\sqrt{X}} = \frac{-1}{\sqrt{-c}} \sin^{-1}\left(\frac{2cx+b}{\sqrt{-q}}\right), \text{ or } \frac{1}{\sqrt{c}} \sinh^{-1}\left(\frac{2cx+b}{\sqrt{q}}\right).$$

$$\int \frac{dx}{x^4+a^4} = \frac{1}{4a^3\sqrt{2}}\left\{\log\left(\frac{x^2+ax\sqrt{2}+a^2}{x^2-ax\sqrt{2}+a^2}\right) + 2\tan^{-1}\left(\frac{ax\sqrt{2}}{a^2-x^2}\right)\right\}.$$

$$\int \frac{dx}{x^4-a^4} = \frac{1}{4a^3}\left\{\log\left(\frac{x-a}{x+a}\right) - 2\tan^{-1}\left(\frac{x}{a}\right)\right\}.$$

Transcendental Functions

$$\int \sin^2 x\, dx = -\tfrac{1}{2}\cos x \sin x + \tfrac{1}{2}x = \tfrac{1}{2}x - \tfrac{1}{4}\sin 2x.$$

$$\int \sin^3 x\, dx = -\tfrac{1}{3}\cos x(\sin^2 x + 2).$$

$$\int \sin^n x\, dx = -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n}\int \sin^{n-2} x\, dx.$$

$$\int \cos x\, dx = \sin x.$$

$$\int \cos^2 x\, dx = \tfrac{1}{2}\sin x \cos x + \tfrac{1}{2}x = \tfrac{1}{2}x + \tfrac{1}{4}\sin 2x.$$

$$\int \cos^3 x\, dx = \tfrac{1}{3}\sin x(\cos^2 x + 2).$$

$$\int \cos^n x\, dx = \frac{1}{n}\cos^{n-1} x \sin x + \frac{n-1}{n}\int \cos^{n-2} x\, dx.$$

$$\int \sin x \cos x\, dx = \tfrac{1}{2}\sin^2 x.$$

$$\int \sin^2 x \cos^2 x\, dx = -\tfrac{1}{8}(\tfrac{1}{4}\sin 4x - x).$$

$$\int \sin x \cos^m x\, dx = -\frac{\cos^{m+1} x}{m+1}.$$

$$\int \sin^m x \cos x \, dx = \frac{\sin^{m+1} x}{m+1}.$$

$$\int \cos^m x \sin^n x \, dx = \frac{\cos^{m-1} x \sin^{n+1} x}{m+n} + \frac{m-1}{m+n} \int \cos^{m-2} x \sin^n x \, dx.$$

$$\int \cos^m x \sin^n x \, dx = -\frac{\sin^{n-1} x \cos^{m+1} x}{m+n} + \frac{n-1}{m+n} \int \cos^m x \sin^{n-2} x \, dx.$$

$$\int \frac{dx}{\cos^n x} = \frac{1}{n-1} \cdot \frac{\sin x}{\cos^{n-1} x} + \frac{n-2}{n-1} \int \frac{dx}{\cos^{n-2} x}.$$

$$\int \tan x \, dx = -\log \cos x.$$

$$\int \tan^2 x \, dx = \tan x - x.$$

$$\int \tan^n x \, dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x \, dx.$$

$$\int \operatorname{ctn} x \, dx = \log \sin x.$$

$$\int \operatorname{ctn}^2 x \, dx = -\operatorname{ctn} x - x.$$

$$\int \operatorname{ctn}^n x \, dx = -\frac{\operatorname{ctn}^{n-1} x}{n-1} - \int \operatorname{ctn}^{n-2} x \, dx.$$

$$\int \sec x \, dx = \log \tan \left(\frac{\pi}{4} + \frac{x}{2} \right) = \tfrac{1}{2} \log \frac{1 + \sin x}{1 - \sin x}.$$

$$\int \sec^2 x \, dx = \tan x.$$

$$\int \sec^n x \, dx = \int \frac{dx}{\cos^n x} = \frac{\sin x}{(n-1)\cos^{n-1} x} + \frac{n-2}{n-1} \int \frac{dx}{\cos^{n-2} x}$$
$$= \frac{\sin x}{(n-1)\cos^{n-1} x} + \frac{n-2}{n-1} \int \sec^{n-2} x \, dx.$$

$$\int \csc x\,dx = \log \tan \tfrac{1}{2} x.$$

$$\int \csc^n x\,dx = \int \frac{dx}{\sin^n x}$$
$$= -\frac{\cos x}{(n-1)\sin^{n-1} x} + \frac{n-2}{n-1}\int \frac{dx}{\sin^{n-2} x}$$
$$= -\frac{\cos x}{(n-1)\sin^{n-1} x} + \frac{n-2}{n-1}\int \csc^{n-2} x\,dx.$$

$$\int \frac{dx}{1+\sin x} = -\tan\left(\tfrac{1}{4}\pi - \tfrac{1}{2}x\right). \quad \text{[See 241.]}$$

$$\int \frac{dx}{1-\sin x} = \operatorname{ctn}\left(\tfrac{1}{4}\pi - \tfrac{1}{2}x\right) = \tan\left(\tfrac{1}{4}\pi + \tfrac{1}{2}x\right).$$

$$\int \frac{dx}{1+\cos x} = \tan \tfrac{1}{2}x, \text{ or } \csc x - \operatorname{ctn} x.$$

$$\int \frac{dx}{1-\cos x} = -\operatorname{ctn} \tfrac{1}{2}x, \text{ or } -\operatorname{ctn} x - \csc x.$$

$$\int \frac{dx}{a \pm b \sin x} = \frac{2 \sec \theta}{a} \cdot \tan^{-1}\left(\sec \theta \cdot \tan \tfrac{1}{2}x \pm \tan \theta\right),$$

if $a > b$, and $b = a \sin \theta$.

$$\int \frac{dx}{a \pm b \sin x} = \frac{\pm \sec \alpha}{b} \log \frac{\sin \frac{1}{2}(\alpha \pm x)}{\cos \frac{1}{2}(x \mp \alpha)},$$

if $b > a$, and $a = b \sin \alpha$. [See 241.]

$$\int \frac{dx}{a + b \cos x} = \frac{-1}{\sqrt{a^2 - b^2}} \cdot \sin^{-1}\left[\frac{b + a \cos x}{a + b \cos x}\right],$$

$$\text{or } \frac{1}{\sqrt{a^2 - b^2}} \sin^{-1}\left[\frac{\sqrt{a^2 - b^2} \cdot \sin x}{a + b \cos x}\right],$$

$$\text{or } \frac{2}{\sqrt{a^2 - b^2}} \tan^{-1}\left[\sqrt{\frac{a-b}{a+b}} \tan \tfrac{1}{2}x\right],$$

$$\text{or } \frac{1}{\sqrt{a^2 - b^2}} \tan^{-1}\left[\frac{\sqrt{a^2 - b^2} \cdot \sin x}{b + a \cos x}\right],$$

$$\int x^m \cos x\,dx = x^m \sin x - m\int x^{m-1}\sin x\,dx.$$

$$\int \frac{\sin x}{x^m}dx = -\frac{1}{m-1}\cdot\frac{\sin x}{x^{m-1}} + \frac{1}{m-1}\int\frac{\cos x}{x^{m-1}}dx.$$

$$\int \frac{\cos x}{x^m}dx = -\frac{1}{m-1}\cdot\frac{\cos x}{x^{m-1}} - \frac{1}{m-1}\int\frac{\sin x}{x^{m-1}}dx.$$

$$\int \frac{\sin x}{x}dx = x - \frac{x^3}{3\cdot 3!} + \frac{x^5}{5\cdot 5!} - \frac{x^7}{7\cdot 7!} + \frac{x^9}{9\cdot 9!}\cdots.$$

$$\int \frac{\cos x}{x}dx = \log x - \frac{x^2}{2\cdot 2!} + \frac{x^4}{4\cdot 4!} - \frac{x^6}{6\cdot 6!} + \frac{x^8}{8\cdot 8!}\cdots.$$

$$\int \frac{x\,dx}{\sin x} = x + \frac{x^3}{3\cdot 3!} + \frac{7x^5}{3\cdot 5\cdot 5!} + \frac{31x^7}{3\cdot 7\cdot 7!} + \frac{127x^9}{3\cdot 5\cdot 9!} + \cdots$$

$$\int \frac{x\,dx}{\cos x} = \frac{x^2}{2} + \frac{x^4}{4\cdot 2!} + \frac{5x^6}{6\cdot 4!} + \frac{61x^8}{8\cdot 6!} + \frac{1385x^{10}}{10\cdot 8!} + \cdots.$$

$$\int \frac{x\,dx}{\sin^2 x} = -x\operatorname{ctn} x + \log\sin x.$$

$$\int \frac{x\,dx}{\cos^2 x} = x\tan x + \log\cos x.$$

$$n^2\int x^m \sin^n x\,dx$$
$$= x^{m-1}\sin^{n-1}x\,(m\sin x - nx\cos x)$$
$$+ n(n-1)\int x^m \sin^{n-2}x\,dx - m(m-1)\int x^{m-2}\sin^n x\,dx.$$

$$n^2\int x^m \cos^n x\,dx$$
$$= x^{m-1}\cos^{n-1}x\,(m\cos x + nx\sin x)$$
$$+ n(n-1)\int x^m \cos^{n-2}x\,dx - m(m-1)\int x^{m-2}\cos^n x\,dx.$$

$$\int\frac{\sin^n x\,dx}{\cos^m x} = \frac{1}{n-m}\left(-\frac{\sin^{n-1}x}{\cos^{m-1}x} + (n-1)\int\frac{\sin^{n-2}x\,dx}{\cos^m x}\right)$$
$$= \frac{1}{m-1}\left(\frac{\sin^{n+1}x}{\cos^{m-1}x} - (n-m+2)\int\frac{\sin^n x\,dx}{\cos^{m-2}x}\right)$$
$$= \frac{1}{m-1}\left(\frac{\sin^{n-1}x}{\cos^{m-1}x} - (n-1)\int\frac{\sin^{n-2}x\,dx}{\cos^{m-2}x}\right).$$

$$\int \frac{\cos^m x\,dx}{\sin^n x} = -\frac{\cos^{m+1} x}{(n-1)\sin^{n-1} x} - \frac{m-n+2}{n-1}\int \frac{\cos^m x\,dx}{\sin^{n-2} x}$$
$$= \frac{\cos^{m-1} x}{(m-n)\sin^{n-1} x} + \frac{m-1}{m-n}\int \frac{\cos^{m-2} x\,dx}{\sin^n x}$$
$$= -\frac{1}{n-1}\,\frac{\cos^{m-1} x}{\sin^{n-1} x} - \frac{m-1}{n-1}\int \frac{\cos^{m-2} x\,dx}{\sin^{n-2} x}.$$

$$\int \frac{\sin^m x\,dx}{\cos^n x} = -\int \frac{\cos^m\left(\frac{\pi}{2} - x\right) d\left(\frac{\pi}{2} - x\right)}{\sin^n\left(\frac{\pi}{2} - x\right)}.$$

$$\int \frac{dx}{\sin x \cos x} = \log \tan x.$$

$$\int \frac{dx}{\cos x \sin^2 x} = \log \tan\left(\frac{\pi}{4} + \frac{x}{2}\right) - \csc x.$$

$$\int \frac{dx}{\sin^m x \cos^n x}$$
$$= \frac{1}{n-1}\cdot\frac{1}{\sin^{m-1} x \cdot \cos^{n-1} x} + \frac{m+n-2}{n-1}\int \frac{dx}{\sin^m x \cdot \cos^{n-2} x}$$
$$= -\frac{1}{m-1}\cdot\frac{1}{\sin^{m-1} x \cdot \cos^{n-1} x} + \frac{m+n-2}{m-1}\int \frac{dx}{\sin^{m-2} x \cdot \cos^n x}.$$

$$\int \frac{dx}{\sin^m x} = -\frac{1}{m-1}\cdot\frac{\cos x}{\sin^{m-1} x} + \frac{m-2}{m-1}\int \frac{dx}{\sin^{m-2} x}.$$

$$\int \frac{x\,dx}{1 + \sin x} = -x \tan \tfrac{1}{2}\left(\tfrac{1}{2}\pi - x\right) + 2 \log \cos \tfrac{1}{2}\left(\tfrac{1}{2}\pi - x\right).$$

$$\int \frac{x\,dx}{1 - \sin x} = x \operatorname{ctn} \tfrac{1}{2}\left(\tfrac{1}{2}\pi - x\right) + 2 \log \sin \tfrac{1}{2}\left(\tfrac{1}{2}\pi - x\right).$$

$$\int \frac{x\,dx}{1 + \cos x} = x \tan \tfrac{1}{2} x + 2 \log \cos \tfrac{1}{2} x.$$

$$\int \frac{x\,dx}{1 - \cos x} = -x \operatorname{ctn} \tfrac{1}{2} x + 2 \log \sin \tfrac{1}{2} x.$$

$$\int \frac{\tan x\,dx}{\sqrt{a + b \tan^2 x}} = \frac{1}{\sqrt{b - a}} \cos^{-1}\left(\frac{\sqrt{b - a}}{\sqrt{b}} \cdot \cos x\right).$$

$$\int \frac{dx}{a + b \tan^2 x} = \frac{1}{a - b}\left[x - \sqrt{\frac{b}{a}} \cdot \tan^{-1}\left(\sqrt{\frac{b}{a}} \cdot \tan x\right)\right].$$

$$\int \frac{\tan x\,dx}{a + b\tan x}$$
$$= \frac{1}{a^2 + b^2}\left\{bx - a\log(a + b\tan x) + a\log\sec x\right\}.$$

$$\int x\sin x\,dx = \sin x - x\cos x.$$

$$\int x^2\sin x\,dx = 2x\sin x - (x^2 - 2)\cos x.$$

$$\int x^3\sin x\,dx = (3x^2 - 6)\sin x - (x^3 - 6x)\cos x.$$

$$\int x^m\sin x\,dx = -x^m\cos x + m\int x^{m-1}\cos x\,dx.$$

$$\int x\cos x\,dx = \cos x + x\sin x.$$

$$\int x^2\cos x\,dx = 2x\cos x + (x^2 - 2)\sin x.$$

$$\int x^3\cos x\,dx = (3x^2 - 6)\cos x + (x^3 - 6x)\sin x.$$

$$\int \frac{x^m\,dx}{\sin^n x}$$
$$= \frac{1}{(n-1)(n-2)}\left[-\frac{x^{m-1}(m\sin x + (n-2)x\cos x)}{\sin^{n-1}x}\right.$$
$$\left.+ (n-2)^2\int \frac{x^m\,dx}{\sin^{n-2}x} + m(m-1)\int \frac{x^{m-2}\,dx}{\sin^{n-2}x}\right].$$

$$\int \frac{x^m\,dx}{\cos^n x}$$
$$= \frac{1}{(n-1)(n-2)}\left[-\frac{x^{m-1}(m\cos x - (n-2)x\sin x)}{\cos^{n-1}x}\right.$$
$$\left.+ (n-2)^2\int \frac{x^m\,dx}{\cos^{n-2}x} + m(m-1)\int \frac{x^{m-2}\,dx}{\cos^{n-2}x}\right].$$

$$\int \frac{\sin^n x\,dx}{x^m}$$
$$= \frac{1}{(m-1)(m-2)}\left[-\frac{\sin^{n-1}x((m-2)\sin x + nx\cos x)}{x^{m-1}}\right.$$
$$\left.- n^2\int \frac{\sin^n x\,dx}{x^{m-2}} + n(n-1)\int \frac{\sin^{n-2}x\,dx}{x^{m-2}}\right].$$

$$\int \frac{\cos^n x\,dx}{x^m}$$

$$= \frac{1}{(m-1)(m-2)}\left[\frac{\cos^{n-1} x\,(nx \cos x - (m-2)\cos x)}{x^{m-1}}\right.$$

$$\left. - n^2 \int \frac{\cos^n x\,dx}{x^{m-2}} + n(n-1)\int \frac{\cos^{n-2} x\,dx}{x^{m-2}}\right].$$

$$\int x^p \sin^m x \cos^n x\,dx$$

$$= \frac{1}{(m+n)^2}\left[x^{p-1} \sin^m x \cos^{n-1} x\,(p \cos x + (m+n) x \sin x)\right.$$

$$+ (n-1)(m+n)\int x^p \sin^m x \cos^{n-2} x\,dx$$

$$- mp \int x^{p-1} \sin^{m-1} x \cos^{n-1} x\,dx$$

$$\left. - p(p-1)\int x^{p-2} \sin^m x \cos^n x\,dx\right].$$

$$= \frac{1}{(m+n)^2}\left[x^{p-1} \sin^{m-1} x \cos^n x\,(p \sin x - (m+n) x \cos x)\right.$$

$$+ (m-1)(m+n)\int x^p \sin^{m-2} x \cos^n x\,dx$$

$$+ np \int x^{p-1} \sin^{m-1} x \cos^{n-1} x\,dx$$

$$\left. - p(p-1)\int x^{p-2} \sin^m x \cos^n x\,dx\right].$$

In this book, we use $\sin^{-1} x$ to denote $\frac{1}{\sin x}$. For this part of the table we use the classical notation $\sin^{-1} x = arcsin(x)$, $\cos^{-1} x = arccos(x)$, etc.

$$\int \sin^{-1} x\,dx = x \sin^{-1} x + \sqrt{1-x^2}.$$

$$\int \cos^{-1} x\,dx = x \cos^{-1} x - \sqrt{1-x^2}.$$

$$\int \tan^{-1} x\,dx = x \tan^{-1} x - \tfrac{1}{2}\log(1+x^2).$$

$$\int \operatorname{ctn}^{-1} x\,dx = x \operatorname{ctn}^{-1} x + \tfrac{1}{2}\log(1+x^2).$$

$$\int \sin mx \sin nx\,dx = \frac{\sin(m-n)x}{2(m-n)} - \frac{\sin(m+n)x}{2(m+n)}.$$

$$\int \sin mx \cos nx\,dx = -\frac{\cos(m-n)x}{2(m-n)} - \frac{\cos(m+n)x}{2(m+n)}.$$

$$\int \cos mx \cos nx\,dx = \frac{\sin(m-n)x}{2(m-n)} + \frac{\sin(m+n)x}{2(m+n)}.$$

$$\int \sin^2 mx\,dx = \frac{1}{2m}(mx - \sin mx \cos mx).$$

$$\int \cos^2 mx\,dx = \frac{1}{2m}(mx + \sin mx \cos mx).$$

$$\int \sin mx \cos mx\,dx = -\frac{1}{4m}\cos 2mx.$$

$$\int \sin nx \sin^m x\,dx = \frac{1}{m+n}\Big[-\cos nx \sin^m x + m\int \cos(n-1)x\cdot\sin^{m-1}x\,dx\Big]$$

$$\int \sec^{-1}x\,dx = x\sec^{-1}x - \log(x+\sqrt{x^2-1}).$$

$$\int \csc^{-1}x\,dx = x\csc^{-1}x + \log(x+\sqrt{x^2-1}).$$

$$\int \operatorname{versin}^{-1}x\,dx = (x-1)\operatorname{versin}^{-1}x + \sqrt{2x-x^2}.$$

$$\int (\sin^{-1}x)^2\,dx = x(\sin^{-1}x)^2 - 2x + 2\sqrt{1-x^2}\sin^{-1}x.$$

$$\int (\cos^{-1}x)^2\,dx = x(\cos^{-1}x)^2 - 2x - 2\sqrt{1-x^2}\cos^{-1}x.$$

$$\int x\sin^{-1}x\,dx = \tfrac{1}{4}\big[(2x^2-1)\sin^{-1}x + x\sqrt{1-x^2}\big].$$

$$\int x\cos^{-1}x\,dx = \tfrac{1}{4}\big[(2x^2-1)\cos^{-1}x - x\sqrt{1-x^2}\big].$$

$$\int x\tan^{-1}x\,dx = \tfrac{1}{2}\big[(x^2+1)\tan^{-1}x - x\big].$$

$$\int x \operatorname{ctn}^{-1} x\,dx = \tfrac{1}{2}[(x^2+1)\operatorname{ctn}^{-1} x + x].$$

$$\int x \sec^{-1} x\,dx = \tfrac{1}{2}[x^2 \sec^{-1} x - \sqrt{x^2-1}].$$

$$\int x \csc^{-1} x\,dx = \tfrac{1}{2}[x^2 \csc^{-1} x + \sqrt{x^2-1}].$$

$$\int x^n \sin^{-1} x\,dx = \frac{1}{n+1}\left(x^{n+1}\sin^{-1} x - \int \frac{x^{n+1}\,dx}{\sqrt{1-x^2}}\right).$$

$$\int x^n \cos^{-1} x\,dx = \frac{1}{n+1}\left(x^{n+1}\cos^{-1} x + \int \frac{x^{n+1}\,dx}{\sqrt{1-x^2}}\right).$$

$$\int x^n \tan^{-1} x\,dx = \frac{1}{n+1}\left(x^{n+1}\tan^{-1} x - \int \frac{x^{n+1}\,dx}{1+x^2}\right).$$

$$\int x^n \operatorname{ctn}^{-1} x\,dx = \frac{1}{n+1}\left(x^{n+1}\operatorname{ctn}^{-1} x + \int \frac{x^{n+1}\,dx}{1+x^2}\right).$$

$$\int \frac{\sin^{-1} x\,dx}{x^2} = \log\left(\frac{1-\sqrt{1-x^2}}{x}\right) - \frac{\sin^{-1} x}{x}.$$

$$\int \frac{\tan^{-1} x\,dx}{x^2} = \log x - \tfrac{1}{2}\log(1+x^2) - \frac{\tan^{-1} x}{x}.$$

$$\int e^{ax}\,dx = \frac{e^{ax}}{a}. \qquad \int f(e^{ax})\,dx = \int \frac{f(y)\,dy}{ay},\ y = e^{ax}.$$

$$\int x e^{ax}\,dx = \frac{e^{ax}}{a^2}(ax-1).$$

$$\int x^m e^{ax}\,dx = \frac{x^m e^{ax}}{a} - \frac{m}{a}\int x^{m-1} e^{ax}\,dx.$$

$$\int \frac{e^{ax}}{x^m}\,dx = \frac{1}{m-1}\left[-\frac{e^{ax}}{x^{m-1}} + a\int \frac{e^{ax}\,dx}{x^{m-1}}\right].$$

$$\int a^{bx}\,dx = \frac{a^{bx}}{b\log a}. \qquad \int f(a^{bx})\,dx = \int \frac{f(y)\,dy}{b\cdot\log a\cdot y},\ y = a^{bx}.$$

$$\int x^n a^x\,dx = \frac{a^x x^n}{\log a} - \frac{n a^x x^{n-1}}{(\log a)^2} + \frac{n(n-1)a^x x^{n-2}}{(\log a)^3} \cdots \pm \frac{n(n-1)(n-2)\cdots 2.1\,a^x}{(\log a)^{n+1}}.$$

$$\int \frac{a^x\,dx}{x^n} = \frac{1}{n-1}\left[-\frac{a^x}{x^{n-1}} - \frac{a^x \cdot \log a}{(n-2)\,x^{n-2}}\right.$$
$$\left. - \frac{a^x \cdot (\log a)^2}{(n-2)(n-3)\,x^{n-3}} - \cdots + \frac{(\log a)^{n-1}}{(n-2)(n-3)\cdots 2.1}\int \frac{a^x\,dx}{x}\right].$$

$$\int \frac{a^x\,dx}{x} = \log x + x \log a + \frac{(x \log a)^2}{2 \cdot 2!} + \frac{(x \log a)^3}{3 \cdot 3!} + \cdots.$$

$$\int \frac{\log x\,dx}{(a+bx)^m}$$
$$= \frac{1}{b(m-1)}\left[-\frac{\log x}{(a+bx)^{m-1}} + \int \frac{dx}{x(a+bx)^{m-1}}\right].$$

$$\int \frac{\log x\,dx}{a+bx} = \frac{1}{b}\log x \cdot \log(a+bx) - \frac{1}{b}\int \frac{\log(a+bx)\,dx}{x}.$$

$$\int (a+bx)\log x\,dx = \frac{(a+bx)^2}{2b}\log x - \frac{a^2 \log x}{2b} - ax - \tfrac{1}{4}bx^2.$$

$$\int \frac{\log x\,dx}{\sqrt{a+bx}}$$
$$= \frac{2}{b}\left[(\log x - 2)\sqrt{a+bx} + \sqrt{a}\,\log(\sqrt{a+bx} + \sqrt{a})\right.$$
$$\left. - \sqrt{a}\,\log(\sqrt{a+bx} - \sqrt{a})\right], \text{ if } a > 0$$
$$= \frac{2}{b}\left[(\log x - 2)\sqrt{a+bx} + 2\sqrt{-a}\,\tan^{-1}\sqrt{\frac{a+bx}{-a}}\right], \text{ if } a < 0.$$

$$\int \sin \log x\,dx = \tfrac{1}{2}x[\sin \log x - \cos \log x].$$

$$\int \cos \log x\,dx = \tfrac{1}{2}x[\sin \log x + \cos \log x].$$

$$\int \frac{(\log x)^n\,dx}{x} = \frac{(\log x)^{n+1}}{n+1}.$$

$$\int \frac{dx}{\log x} = \log(\log x) + \log x + \frac{(\log x)^2}{2 \cdot 2!} + \frac{(\log x)^3}{3 \cdot 3!} + \cdots.$$

$$\int \frac{dx}{(\log x)^n} = -\frac{x}{(n-1)(\log x)^{n-1}} + \frac{1}{n-1}\int \frac{dx}{(\log x)^{n-1}}.$$

$$\int \frac{x^m\,dx}{(\log x)^n} = -\frac{x^{m+1}}{(n-1)(\log x)^{n-1}} + \frac{m+1}{n-1}\int \frac{x^m\,dx}{(\log x)^{n-1}}.$$

$$\int \log x\,dx = x\log x - x.$$

$$\int x^m \log x\,dx = x^{m+1}\left[\frac{\log x}{m+1} - \frac{1}{(m+1)^2}\right].$$

$$\int (\log x)^n\,dx = x(\log x)^n - n\int (\log x)^{n-1}\,dx.$$

$$\int x^m(\log x)^n\,dx = \frac{x^{m+1}(\log x)^n}{m+1} - \frac{n}{m+1}\int x^m(\log x)^{n-1}\,dx.$$

$$\int \frac{dx}{x\log x} = \log(\log x), \text{ and } \int \frac{(n-1)\,dx}{x(\log x)^n} = \frac{-1}{(\log x)^{n-1}}.$$

$$\int \log(a^2+x^2)\,dx = x\cdot\log(a^2+x^2) - 2x + 2a\cdot\tan^{-1}\left(\frac{x}{a}\right).$$

$$\int \frac{dx}{1+e^x} = \log\frac{e^x}{1+e^x}.$$

$$\int \frac{dx}{a+be^{mx}} = \frac{1}{am}[mx - \log(a+be^{mx})].$$

$$\int \frac{dx}{ae^{mx}+be^{-mx}} = \frac{1}{m\sqrt{ab}}\tan^{-1}\left(e^{mx}\sqrt{\frac{a}{b}}\right).$$

$$\int \frac{dx}{\sqrt{a+be^{mx}}} = \frac{1}{m\sqrt{a}}\{\log(\sqrt{a+be^{mx}}-\sqrt{a})$$

$$-\log(\sqrt{a+b\,e^{mx}}+\sqrt{a})\}, \text{ or } \frac{2}{m\sqrt{-a}}\tan^{-1}\frac{\sqrt{a+be^{mx}}}{\sqrt{-a}}.$$

$$\int \frac{xe^x\,dx}{(1+x)^2} = \frac{e^x}{1+x}, \quad \int x^n\cdot e^{ax^{n+1}}dx = \frac{e^{ax^{n+1}}}{a(n+1)}.$$

$$\int e^{ax}\sin px\,dx = \frac{e^{ax}(a\sin px - p\cos px)}{a^2+p^2}.$$

$$\int e^{ax}\cos px\,dx = \frac{e^{ax}(a\cos px + p\sin px)}{a^2+p^2}.$$

$$\int e^{ax}\log x\,dx = \frac{e^{ax}\log x}{a} - \frac{1}{a}\int \frac{e^{ax}\,dx}{x}.$$

$$\int e^{ax}\sin^2 x\,dx = \frac{e^{ax}}{4+a^2}\left(\sin x(a\sin x - 2\cos x) + \frac{2}{a}\right).$$

$$\int e^{ax}\cos^2 x\,dx = \frac{e^{ax}}{4+a^2}\left(\cos x\,(2\sin x + a\cos x) + \frac{2}{a}\right).$$

$$\int e^{ax}\sin^n bx\,dx = \frac{1}{a^2+n^2b^2}\Big((a\sin bx$$

$$-\,nb\cos bx)\,e^{ax}\sin^{n-1}bx + n(n-1)\,b^2\int e^{ax}\sin^{n-2}bx\cdot dx\Big).$$

$$\int e^{ax}\cos^n bx\,dx = \frac{1}{a^2+n^2b^2}\Big((a\cos bx$$

$$+\,nb\sin bx)\,e^{ax}\cos^{n-1}bx + n(n-1)\,b^2\int e^{ax}\cos^{n-2}bx\,dx\Big).$$

$$\int e^{ax}\tan^n x\,dx$$

$$= \frac{e^{ax}\tan^{n-1}x}{n-1} - \frac{a}{n-1}\int e^{ax}\tan^{n-1}x\,dx - \int e^{ax}\tan^{n-2}x\,dx.$$

$$\int e^{ax}\operatorname{ctn}^n x\,dx$$

$$= -\frac{e^{ax}\operatorname{ctn}^{n-1}x}{n-1} + \frac{a}{n-1}\int e^{ax}\operatorname{ctn}^{n-1}x\,dx - \int e^{ax}\operatorname{ctn}^{n-2}x\,dx.$$

$$\int \frac{e^{ax}\,dx}{\sin^n x} = -\,e^{ax}\frac{a\sin x + (n-2)\cos x}{(n-1)(n-2)\sin^{n-1}x}$$

$$+\,\frac{a^2+(n-2)^2}{(n-1)(n-2)}\int\frac{e^{ax}\,dx}{\sin^{n-2}x}.$$

$$\int \frac{e^{ax}\,dx}{\cos^n x} = -\,e^{ax}\frac{a\cos x - (n-2)\sin x}{(n-1)(n-2)\cos^{n-1}x}$$

$$+\,\frac{a^2+(n-2)^2}{(n-1)(n-2)}\int\frac{e^{ax}\,dx}{\cos^{n-2}x}.$$

$$\int e^{ax}\sin^m x\cos^n x\,dx$$

$$= \frac{1}{(m+n)^2+a^2}\Big\{e^{ax}\sin^{mx}x\cos^{n-1}x\,(a\cos x + (m+n)\sin x)$$

$$-\,ma\int e^{ax}\sin^{m-1}x\cos^{m-1}x\,dx$$

$$+\,(n-1)(m+n)\int e^{ax}\sin^m x\cos^{n-2}x\,dx\Big\}$$

$$\int \frac{x^m\,dx}{\log x} = \int \frac{e^{-y}}{y}\,dy, \text{ where } y = -(m+1)\log x.$$

MISCELLANEOUS DEFINITE INTEGRALS

$$\int_0^\infty \frac{a\,dx}{a^2+x^2} = \frac{\pi}{2}, \text{ if } a > 0;\ 0, \text{ if } a = 0;\ -\frac{\pi}{2}, \text{ if } a < 0.$$

$$\int_0^\infty x^{n-1} e^{-x}\,dx = \int_0^1 \left[\log \frac{1}{x}\right]^{n-1} dx \equiv \Gamma(n).$$

$$\Gamma(z+1) = z \cdot \Gamma(z), \text{ if } z > 0.$$

$$\Gamma(y)\cdot\Gamma(1-y) = \frac{\pi}{\sin \pi y}, \text{ if } 1 > y > 0. \quad \Gamma(2) = \Gamma(1) = 1.$$

$$\Gamma(n+1) = n!, \text{ if } n \text{ is an integer.} \qquad \Gamma(z) = \Pi(z-1).$$

$$\Gamma(\tfrac{1}{2}) = \sqrt{\pi}. \qquad Z(y) = D_y[\log \Gamma(y)]. \quad Z(1) = -0.577216.$$

$$\int_0^1 x^{m-1}(1-x)^{n-1}\,dx = \int_0^\infty \frac{x^{m-1}\,dx}{(1+x)^{m+n}} = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}.$$

$$\int_0^{\frac{\pi}{2}} \sin^n x\,dx = \int_0^{\frac{\pi}{2}} \cos^n x\,dx$$

$$= \frac{1\cdot 3\cdot 5 \cdots (n-1)}{2\cdot 4\cdot 6 \cdots (n)} \cdot \frac{\pi}{2}, \text{ if } n \text{ is an even integer,}$$

$$= \frac{2\cdot 4\cdot 6 \cdots (n-1)}{1\cdot 3\cdot 5\cdot 7 \cdots n}, \text{ if } n \text{ is an odd integer,}$$

$$= \tfrac{1}{2}\sqrt{\pi}\,\frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}+1\right)}, \text{ for any value of } n \text{ greater than } -1.$$

$$\int_0^\infty \frac{\sin mx\,dx}{x} = \frac{\pi}{2}, \text{ if } m > 0;\ 0, \text{ if } m = 0;\ -\frac{\pi}{2}, \text{ if } m < 0.$$

$$\int_0^1 \log\left(\frac{1+x}{1-x}\right)\cdot\frac{dx}{x} = \frac{\pi^2}{4}.$$

$$\int_0^1 \frac{\log x\,dx}{\sqrt{1-x^2}} = -\frac{\pi}{2}\log 2.$$

$$\int_0^1 \frac{(x^p - x^q)\,dx}{\log x} = \log\frac{p+1}{q+1}, \text{ if } p+1 > 0,\ q+1 > 0.$$

$$\int_0^1 (\log x)^n\,dx = (-1)^n \cdot n!.$$

$$\int_0^1 \left(\log\frac{1}{x}\right)^{\frac{1}{2}} dx = \frac{\sqrt{\pi}}{2}.$$

$$\int_0^1 \left(\log\frac{1}{x}\right)^n dx = n!.$$

$$\int_0^1 \frac{dx}{\sqrt{\log\left(\frac{1}{x}\right)}} = \sqrt{\pi}.$$

$$\int_0^1 x^m \log\left(\frac{1}{x}\right)^n dx = \frac{\Gamma(n+1)}{(m+1)^{n+1}}, \text{ if } m+1 > 0,\ n+1 > 0.$$

$$\int_0^\infty \log\left(\frac{e^x+1}{e^x-1}\right) dx = \frac{\pi^2}{4}.$$

$$\int_0^{\frac{\pi}{2}} \log\sin x\,dx = \int_0^{\frac{\pi}{2}} \log\cos x\,dx = -\frac{\pi}{2}\cdot\log 2.$$

$$\int_0^\pi x\cdot\log\sin x\,dx = -\frac{\pi^2}{2}\log 2.$$

$$\int_0^\pi \log(a \pm b\cos x)\,dx = \pi\log\left(\frac{a+\sqrt{a^2-b^2}}{2}\right). \qquad a \geqq b.$$

$$\int_0^\infty \frac{dx}{e^{nx}+e^{-nx}} = \frac{\pi}{4n}.$$

$$\int_0^\infty \frac{x\,dx}{e^{nx}-e^{-nx}} = \frac{\pi^2}{8n^2}.$$

$$\int_0^{\pi i} \sinh(mx)\cdot\sinh(nx)\,dx = \int_0^{\pi i} \cosh(mx)\cdot\cosh(nx)\,dx$$
$$= 0, \text{ if } m \text{ is different from } n.$$

$$\int_0^{\pi i} \cosh^2(mx)\,dx = -\int_0^{\pi i} \sinh^2(mx)\,dx = \frac{\pi i}{2}.$$

$$\int_{-\pi i}^{+\pi i} \sinh(mx)\,dx = 0.$$

$$\int_0^{\pi i} \cosh(mx)\,dx = 0.$$

$$\int_{-\pi i}^{\pi i} \sinh(mx)\cosh(nx)\,dx = 0.$$

$$\int_0^{\pi i} \sinh(mx)\cosh(mx)\,dx = 0.$$

$$\int_0^{\infty} e^{-ax}\cos mx\,dx = \frac{a}{a^2+m^2}, \text{ if } a > 0.$$

$$\int_0^{\infty} e^{-ax}\sin mx\,dx = \frac{m}{a^2+m^2}, \text{ if } a > 0.$$

$$\int_0^{\infty} e^{-a^2x^2}\cos bx\,dx = \frac{\sqrt{\pi}\cdot e^{-\frac{b^2}{4a^2}}}{2a}. \qquad a > 0.$$

$$\int_0^1 \frac{\log x}{1-x}\,dx = -\frac{\pi^2}{6}.$$

$$\int_0^1 \frac{\log x}{1+x}\,dx = -\frac{\pi^2}{12}.$$

$$\int_0^1 \frac{\log x}{1-x^2}\,dx = -\frac{\pi^2}{8}.$$

$$\int_0^{\infty} \frac{\sin x\cdot\cos mx\,dx}{x} = 0, \text{ if } m < -1 \text{ or } m > 1;$$

$$\frac{\pi}{4}, \text{ if } m = -1 \text{ or } m = 1; \ \frac{\pi}{2}, \text{ if } -1 < m < 1.$$

$$\int_0^{\infty} \frac{\sin^2 x\,dx}{x^2} = \frac{\pi}{2}.$$

$$\int_0^{\infty} \cos(x^2)\,dx = \int_0^{\infty} \sin(x^2)\,dx = \tfrac{1}{2}\sqrt{\frac{\pi}{2}}.$$

$$\int_0^{\pi} \sin kx\cdot\sin mx\,dx = \int_0^{\pi} \cos kx\cdot\cos mx\,dx = 0,$$

if k is different from m.

$$\int_0^{\pi} \sin^2 mx\,dx = \int_0^{\pi} \cos^2 mx\,dx = \frac{\pi}{2}.$$

$$\int_0^{\infty} \frac{\cos mx\,dx}{1+x^2} = \frac{\pi}{2}\cdot e^{-m}. \qquad m > 0.$$

$$\int_0^{\infty} \frac{\cos x\,dx}{\sqrt{x}} = \int_0^{\infty} \frac{\sin x\,dx}{\sqrt{x}} = \sqrt{\frac{\pi}{2}}.$$

$$\int_0^\infty e^{-a^2x^2}\,dx = \frac{1}{2a}\sqrt{\pi}\cdot = \frac{1}{2a}\,\Gamma(\tfrac{1}{2}).$$

$$\int_0^\infty x^n e^{-ax}\,dx = \frac{\Gamma(n+1)}{a^{n+1}} = \frac{n!}{a^{n+1}}.$$

$$\int_0^\infty x^{2n} e^{-ax^2}\,dx = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^{n+1}a^n}\sqrt{\frac{\pi}{a}}.$$

$$\int_0^\infty e^{-x^2-\frac{a^2}{x^2}}\,dx = \frac{e^{-2a}\sqrt{\pi}}{2}. \qquad a > 0.$$

$$\int_0^\infty e^{-nx}\sqrt{x}\,dx = \frac{1}{2n}\sqrt{\frac{\pi}{n}}.$$

$$\int_0^\infty \frac{e^{-nx}}{\sqrt{x}}\,dx = \sqrt{\frac{\pi}{n}}. \qquad a > 0.$$

TRIGONOMETRIC FUNCTIONS

	0°.	30°.	45°.	60°.	90°.	120°.	135°.	150°.	180°.
sin	0	$\frac{1}{2}$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{3}$	1	$\frac{1}{2}\sqrt{3}$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}$	0
cos	1	$\frac{1}{2}\sqrt{3}$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{2}\sqrt{2}$	$-\frac{1}{2}\sqrt{3}$	-1
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞	$-\sqrt{3}$	-1	$-\frac{1}{\sqrt{3}}$	0
ctn	∞	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0	$-\frac{1}{\sqrt{3}}$	-1	$-\sqrt{3}$	∞
sec	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	∞	-2	$-\sqrt{2}$	$-\frac{2}{\sqrt{3}}$	-1
csc	∞	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	∞

$$\sin\tfrac{1}{2}a = \sqrt{\tfrac{1}{2}(1-\cos a)}.$$

$$\cos\tfrac{1}{2}a = \sqrt{\tfrac{1}{2}(1+\cos a)}.$$

$$\tan\tfrac{1}{2}a = \sqrt{\frac{1-\cos a}{1+\cos a}} = \frac{1-\cos a}{\sin a} = \frac{\sin a}{1+\cos a}.$$

$$\sin 2a = 2\sin a\cos a.$$

$$\sin 3a = 3\sin a - 4\sin^3 a.$$

$$\sin 4a = 8\cos^3 a\cdot\sin a - 4\cos a\sin a.$$

$\sin 5a = 5\sin a - 20\sin^3 a + 16\sin^5 a.$

$\sin 6a = 32\cos^5 a\sin a - 32\cos^3 a\sin a + 6\cos a\sin a.$

$\cos 2a = \cos^2 a - \sin^2 a = 1 - 2\sin^2 a = 2\cos^2 a - 1.$

$\cos 3a = 4\cos^3 a - 3\cos a.$

$\cos 4a = 8\cos^4 a - 8\cos^2 a + 1.$

$\cos 5a = 16\cos^5 a - 20\cos^3 a + 5\cos a.$

$\cos 6a = 32\cos^6 a - 48\cos^4 a + 18\cos^2 a - 1.$

$$\tan 2a = \frac{2\tan a}{1 - \tan^2 a}.$$

$$\operatorname{ctn} 2a = \frac{\operatorname{ctn}^2 a - 1}{2\operatorname{ctn} a}.$$

$\sin(a \pm \beta) = \sin a \cdot \cos\beta \pm \cos a \cdot \sin\beta.$

$\cos(a \pm \beta) = \cos a \cdot \cos\beta \mp \sin a \cdot \sin\beta.$

$$\tan(a \pm \beta) = \frac{\tan a \pm \tan\beta}{1 \mp \tan a \cdot \tan\beta}.$$

$$\operatorname{ctn}(a \pm \beta) = \frac{\operatorname{ctn} a \cdot \operatorname{ctn}\beta \mp 1}{\operatorname{ctn} a \pm \operatorname{ctn}\beta}.$$

$\sin a \pm \sin\beta = 2\sin\tfrac{1}{2}(a \pm \beta)\cdot\cos\tfrac{1}{2}(a \mp \beta).$

$\cos a + \cos\beta = 2\cos\tfrac{1}{2}(a + \beta)\cdot\cos\tfrac{1}{2}(a - \beta).$

$\cos a - \cos\beta = -2\sin\tfrac{1}{2}(a + \beta)\cdot\sin\tfrac{1}{2}(a - \beta).$

$$\tan a \pm \tan\beta = \frac{\sin(a \pm \beta)}{\cos a \cdot \cos\beta}.$$

$$\operatorname{ctn} a \pm \operatorname{ctn}\beta = \pm\frac{\sin(a \pm \beta)}{\sin a \cdot \sin\beta}.$$

$\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha + \beta) + \sin(\alpha - \beta)]$

$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha + \beta) + \cos(\alpha - \beta)]$

$\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha - \beta) - \cos(\alpha + \beta)]$

HYPERBOLIC FUNCTIONS

$$\sinh x = \tfrac{1}{2}(e^{x} - e^{-x}) = -\sinh(-x) = -i\sin(ix)$$
$$= (\operatorname{csch} x)^{-1} = 2\tanh\tfrac{1}{2}x \div (1 - \tanh^{2}\tfrac{1}{2}x).$$

$$\cosh x = \tfrac{1}{2}(e^{x} + e^{-x}) = \cosh(-x) = \cos(ix) = (\operatorname{sech} x)^{-1}$$
$$= (1 + \tanh^{2}\tfrac{1}{2}x) \div (1 - \tanh^{2}\tfrac{1}{2}x).$$

$$\tanh x = (e^{x} - e^{-x}) \div (e^{x} + e^{-x}) = -\tanh(-x)$$
$$= -i\tan(ix) = (\operatorname{ctnh} x)^{-1} = \sinh x \div \cosh x.$$

$$\cosh xi = \cos x.$$

$$\sinh xi = i\sin x.$$

$$\cosh^{2}x - \sinh^{2}x = 1.$$

$$1 - \tanh^{2}x = \operatorname{sech}^{2}x.$$

$$1 - \operatorname{ctnh}^{2}x = -\operatorname{csch}^{2}x.$$

$$\sinh(x \pm y) = \sinh x \cdot \cosh y \pm \cosh x \cdot \sinh y.$$

$$\cosh(x \pm y) = \cosh x \cdot \cosh y \pm \sinh x \cdot \sinh y.$$

$$\tanh(x \pm y) = (\tanh x \pm \tanh y) \div (1 \pm \tanh x \cdot \tanh y).$$

$$\sinh(2x) = 2\sinh x\cosh x.$$

$$\cosh(2x) = \cosh^{2}x + \sinh^{2}x = 2\cosh^{2}x - 1 = 1 + 2\sinh^{2}x.$$

$$\tanh(2x) = 2\tanh x \div (1 + \tanh^{2}x).$$

$$\sinh(\tfrac{1}{2}x) = \sqrt{\tfrac{1}{2}(\cosh x - 1)}.$$

$$\cosh(\tfrac{1}{2}x) = \sqrt{\tfrac{1}{2}(\cosh x + 1)}.$$

$$\tanh(\tfrac{1}{2}x) = (\cosh x - 1) \div \sinh x = \sinh x \div (\cosh x + 1).$$

$$\sinh x + \sinh y = 2\sinh\tfrac{1}{2}(x + y) \cdot \cosh\tfrac{1}{2}(x - y).$$

$$\sinh x - \sinh y = 2\cosh\tfrac{1}{2}(x + y) \cdot \sinh\tfrac{1}{2}(x - y).$$

$$\frac{\sin\alpha \pm \sin\beta}{\cos\alpha + \cos\beta} = \tan\tfrac{1}{2}(\alpha \pm \beta).$$

$$\frac{\sin\alpha \pm \sin\beta}{\cos\alpha - \cos\beta} = -\operatorname{ctn}\tfrac{1}{2}(\alpha \mp \beta).$$

$$\frac{\sin\alpha + \sin\beta}{\sin\alpha - \sin\beta} = \frac{\tan\frac{1}{2}(\alpha + \beta)}{\tan\frac{1}{2}(\alpha - \beta)}.$$

$$\sin^2 \alpha - \sin^2 \beta = \sin(\alpha + \beta) \cdot \sin(\alpha - \beta).$$

$$\cos^2 \alpha - \cos^2 \beta = -\sin(\alpha + \beta) \cdot \sin(\alpha - \beta).$$

$$\cos^2 \alpha - \sin^2 \beta = \cos(\alpha + \beta) \cdot \cos(\alpha - \beta).$$

$$\sin xi = \tfrac{1}{2} i (e^x - e^{-x}) = i \sinh x.$$

$$\cos xi = \tfrac{1}{2} (e^x + e^{-x}) = \cosh x.$$

$$\tan xi = \frac{i(e^x - e^{-x})}{e^x + e^{-x}} = i \tanh x.$$

$$e^{x+yi} = e^x \cos y + i e^x \sin y.$$

$$a^{x+yi} = a^x \cos(y \cdot \log a) + i a^x \sin(y \cdot \log a).$$

$$(\cos\theta \pm i \cdot \sin\theta)^n = \cos n\theta \pm i \cdot \sin n\theta.$$

$$\sin x = -\tfrac{1}{2} i (e^{xi} - e^{-xi}).$$

$$\cos x = \tfrac{1}{2} (e^{xi} + e^{-xi}).$$

$$\tan x = -i \frac{e^{2xi} - 1}{e^{2xi} + 1}.$$

$$\sin(x \pm yi) = \sin x \cos yi \pm \cos x \sin yi$$
$$= \sin x \cosh y \pm i \cos x \sinh y.$$

$$\cos(x \pm yi) = \cos x \cos yi \mp \sin x \sin yi$$
$$= \cos x \cosh y \mp i \sin x \sinh y.$$

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

$$a^2 = b^2 + c^2 - 2bc \cos A.$$

$$\frac{a+b}{a-b} = \frac{\sin A + \sin B}{\sin A - \sin B} = \frac{\tan\frac{1}{2}(A+B)}{\tan\frac{1}{2}(A-B)} = \frac{\operatorname{ctn}\frac{1}{2}C}{\tan\frac{1}{2}(A-B)}.$$

$$\sin\tfrac{1}{2}A = \sqrt{\frac{(s-b)(s-c)}{bc}}, \text{ where } 2s = a + b + c.$$

$$\cos\tfrac{1}{2}A = \sqrt{\frac{s(s-a)}{bc}}.$$

$$\tan\tfrac{1}{2}A = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}.$$

$$\text{Area} = \tfrac{1}{2} bc \sin A = \sqrt{s(s-a)(s-b)(s-c)}.$$

$$\tan^{-1}x = \sin^{-1}\frac{x}{\sqrt{1+x^2}} = \cos^{-1}\frac{1}{\sqrt{1+x^2}} = \tfrac{1}{2}\sin^{-1}\frac{2x}{1+x^2}$$

$$= \operatorname{ctn}^{-1}\frac{1}{x} = \tfrac{1}{2}\pi - \operatorname{ctn}^{-1}x = \sec^{-1}\sqrt{1+x^2}$$

$$= \tfrac{1}{2}\pi - \tan^{-1}\frac{1}{x}$$

$$= \csc^{-1}\frac{\sqrt{1+x^2}}{x} = \tfrac{1}{2}\cos^{-1}\left[\frac{1-x^2}{1+x^2}\right]$$

$$= 2\cos^{-1}\left[\frac{1+\sqrt{1+x^2}}{2\sqrt{1+x^2}}\right]^{\frac{1}{2}} = 2\sin^{-1}\left[\frac{\sqrt{1+x^2}-1}{2\sqrt{1+x^2}}\right]^{\frac{1}{2}}$$

$$= \tfrac{1}{2}\tan^{-1}\frac{2x}{1-x^2} = 2\tan^{-1}\left[\frac{\sqrt{1+x^2}-1}{x}\right]$$

$$= -\tan^{-1}c + \tan^{-1}\left[\frac{x+c}{1-cx}\right] = -\tan^{-1}(-x)$$

$$= \tfrac{1}{2}i\log\frac{1-xi}{1+xi} = \tfrac{1}{2}i\log\frac{i+x}{i-x}$$

$$= -\tfrac{1}{2}i\log\frac{1+xi}{1-xi}.$$