PART IV
Generating Functions

Suppose we are given a sequence $a_0, a_1, \ldots$. The "ordinary generating function" associated with the sequence $a_0, a_1, \ldots$ is the function $A(x)$ whose value at x is the power series $\sum_{i\ge 0} a_i x^i$. In other words, "ordinary generating function of" is a map (function) from sequences to power series that "packages" the entire series of numbers $a_0, a_1, \ldots$ into a single function $A(x)$. Generating functions are not limited to sequences with single indices; for example, we will see that

$$\sum_{n\ge 0}\sum_{k\ge 0}\binom{n}{k}x^k y^n \;=\; (1-y(1+x))^{-1}.$$

For simplicity, this introductory discussion is phrased in terms of singly indexed sequences.

It is often easier to manipulate generating functions than it is to manipulate sequences $a_0, a_1, \ldots$. We can obtain information about a sequence through the manipulation of its generating function. We have seen a little of this in Example 1.14 (p. 19) and in Section 1.5 (p. 36). In the next two chapters, we will study generating functions in more detail.

What sort of information is needed about a sequence $a_0, a_1, \ldots$, to obtain $A(x)$, the generating function for the sequence? Of course, an explicit formula for a_n as a function of n would provide this but there are often better methods:

- Recursions: A recursion for the a_n's may give an equation that can be solved for $A(x)$. We'll study this in Sections 10.2 and 11.1.
- Constructions: A method for constructing the objects counted by the a_n's may lead to an equation for $A(x)$. Sometimes this can be done using an extension of the Rules of Sum and Product. See Sections 10.4 and 11.2.

Given a generating function $A(x)$, what sort of information might we obtain about the sequence $a_0, a_1, \ldots$ associated with it?

- An explicit formula: *Taylor's Theorem* from calculus is an important tool. It tells us that $A(x) = \sum\big(A^{(n)}(0)/n!\big)x^n$, and so gives us a formula if $A(x)$ is simple enough. In particular, we easily obtain a rather simple formula for b_n, the number of unlabeled binary trees with n leaves. In Section 9.3 we obtained only a recursion.
- A recursion: Simply equate coefficients in an equation that involves $A(x)$. This is done in various places in the following chapters, but especially in Section 10.3.
- Statistical information: For example, the expected number of cycles in a random permutation of $\underline{n}$ is approximately $\ln n$. The degree of the root of a random labeled RP-tree is approximately one more than a "Poisson random variable," a subject beyond the scope of this text.
- Asymptotic information: In other words, how does a_n behave when n is large? Methods for doing this are discussed in Section 11.4.
- Prove identities: We saw some of this in Section 1.5 (p. 36).

We hope that the preceding list has convinced you that generating functions are an important tool that yield results that are sometimes difficult (or perhaps even impossible) to obtain by other methods. If you find generating functions difficult to understand, keep this motivation in mind as you study them.

The next chapter introduces the basic concepts associated with generating functions:

- Basic concepts: We define a generating function and look at some basic manipulations.
- Recursions: If you have encountered recursions in other courses or do not wish to study them, you can skim Section 10.2. However, if you had difficulty in Section 10.1, you should use this section to obtain additional practice with generating functions.
- Manipulations: In Section 10.3 we present some techniques for manipulating generating functions.
- Rule of Sum and Product: We consider Section 10.4 to be the heart of this chapter. In it, we extend the definition of generating function a bit and obtain Rules of Sum and Product which do for generating functions what the rules with the same name did for basic counting in Chapter 1.

In Chapter 11 we take up four separate topics:

- Systems of recursions: This is a continuation of the discussion in Section 10.2.
- Exponential generating functions: These play the same role for objects with labels that ordinary generating functions play for unlabeled objects in Section 10.4.
- Counting objects with symmetries: We apply generating functions to the problems discussed in Section 4.3 (p. 111).
- Asymptotics: We discuss methods for obtaining asymptotics from generating functions and, to a lesser extent, from recursions.

The sections in Chapter 11 can be read independently of one another; however, some of the asymptotic examples make use of results (but not methods) from Section 11.2.