Recursive projections of symmetric tensors and Marcus's proof of the Schur inequality

S. Gill Williamson[1]

Abstract

In a 1918 paper [Sch18], Schur proved a remarkable inequality that related group representations, Hermitian forms and determinants. He also gave concise necessary and sufficient conditions for equality. In [Mar64], Marcus gave a beautiful short proof of Schur's inequality by applying the Cauchy-Schwarz inequality to symmetric tensors, but he did not discuss the case of equality. In [Wil69], Williamson gave an inductive proof of Schur's equality conditions by contracting Marcus's symmetric tensors onto lower dimensional subspaces where they remained symmetric tensors. Here we unify these results notationally and conceptually, replacing contraction operators with the more geometrically intuitive projection operators.

1. Introduction

The following theorem will be the focus of this paper.

Theorem 1.1 (Schur's theorem for finite groups and hermitian forms). *Let G be a subgroup of the symmetric group S_n on $\underline{n} = \{1, 2, \dots, n\}$. Let $H = (h_{ij})$ be an $n \times n$ complex positive definite Hermitian matrix and define $\mathbb{G}_H$ to be the group generated by all transpositions (i, j) such that $h_{ij} \neq 0$. Let M be a representation of G as unitary linear operators on U, $\dim(U) = m$, and let $M_H = \sum_{\sigma\in G} M(\sigma) \prod_{i=1}^{n} h_{i\sigma(i)}$. M_H, called a generalized matrix function, is positive definite Hermitian, and for $u \in U$, $\|u\| = 1$,*

$$\det(H) \leq (M_H u, u) \tag{1.2}$$

$$\det(H) = (M_H u, u) \text{ if and only if } \mathbb{G}_H \subseteq G \text{ and} \tag{1.3}$$

$$(M(\sigma)u, u) = \epsilon(\sigma) \text{ for all } \sigma \in \mathbb{G}_H \text{ where } \epsilon(\sigma) \text{ is the sign of } \sigma.$$

[1] http://cseweb.ucsd.edu/~gill

In this section we discuss the basics. In Section 2 we prove Marcus's generalization of Schur's inequality. In Section 3 we treat the case of equality. In Section 4 we discuss the combinatorial lemmas needed for Section 3. In Section 5 we give examples of Schur's inequality and discuss the trace version.

Remark 1.4 (**Comments on theorem 1.1**).

(**1**) Throughout the rest of the paper, $\mathbf{M}_n(\mathbb{C})$ denotes the $n\times n$ matrices with entries in $\mathbb{C}$, the complex numbers. Likewise, the order (or cardinality) of the finite group G will be denoted by g.

(**2**) Referring to equality condition 1.3 we have

$$M_H = \sum_{\sigma\in G} M(\sigma)\prod_{i=1}^{n} h_{i\sigma(i)} \text{ implies that } (M_H u, u) = \sum_{\sigma\in G}(M(\sigma)u,u)\prod_{i=1}^{n} h_{i\sigma(i)}.$$

Assume $M(\sigma_1)$ and $M(\sigma_2)$ unitary and $u\in U$ has $\|u\|=1$. If $|(M(\sigma_1)u,u)| = |(M(\sigma_2)u,u)| = 1$ then u is a unit eigenvector of both $M(\sigma_1)$ and $M(\sigma_2)$. We have $M(\sigma_j)u=\lambda_j u$, $\lambda_j = e^{ir_j}$, $i=1,2$. Thus, $(M(\sigma_1)u,u)(M(\sigma_2)u,u) = \lambda_1\lambda_2 = (M(\sigma_1)M(\sigma_2)u,u) = (M(\sigma_1\sigma_2)u,u)$.

(**3**) A group $\mathbb{G}_D$, analogous to $\mathbb{G}_H$ of theorem 1.1, can defined for any $D\in\mathbf{M}_n(\mathbb{C})$. Let $K\subseteq\{1,2,\ldots,n\}$ be an orbit of $\mathbb{G}_D$. Since $\mathbb{G}_D$ is generated by transpositions, the restriction $\mathbb{G}_D|K$ is the symmetric group S_K.

Remark 1.5 (**Inner products on tensors**). Let $\underline{n}=\{1,\ldots,n\}$. Denote by $\underline{n}^{\underline{n}}$ the set of all functions from $\underline{n}$ to $\underline{n}$. The notation $\Gamma_n\equiv\underline{n}^{\underline{n}}$ is also common in this subject. Note the cardinality, $|\Gamma_n| = |\underline{n}^{\underline{n}}| = n^n$. Let $e_1,\ldots,e_n$ be an orthonormal basis for the unitary space V. Using the inner product in U, we have an inner product on $U\otimes[\otimes^n V]$ which on homogeneous elements is

$$(u_1\otimes x_1\otimes\cdots\otimes x_n, u_2\otimes y_1\otimes\cdots\otimes y_n) = (u_1,u_2)\prod_{i=1}^{n}(x_i,y_i).$$

Note that

$$\{e_\alpha \mid e_\alpha = e_{\alpha(1)}\otimes\cdots\otimes e_{\alpha(n)},\ \alpha\in\underline{n}^{\underline{n}}\} \tag{1.6}$$

is an orthonormal basis for $\otimes^n V$.

Definition 1.7 (**Generalized symmetry operators).** Let G be a subgroup of the symmetric group S_n on $\{1,2,\ldots,n\}$. Let M be a representation of G as unitary linear operators on a unitary space U, $\dim U = m$. Define an endomorphism of $U\otimes[\otimes^n V]$ by $T_G = \sum_{\sigma\in G} M(\sigma)\otimes P(\sigma)$ where $P(\sigma)$ is the permutation operator defined by $P(\sigma)(e_\alpha) = e_{\alpha(\sigma^{-1}(1))}\otimes\cdots\otimes$

$e_{\alpha(\sigma^{-1}(n))}$ on the basis and extended (conjugate) linearly to $\otimes^n V$. On homogeneous tensors

(1.8) $$P(\sigma)(x_1 \otimes \cdots \otimes x_n) = x_{\sigma^{-1}(1)} \otimes \cdots \otimes x_{\sigma^{-1}(n)}.$$

T_G will be called a *generalized symmetry operator* on $U \otimes (\otimes^n V)$ of *degree* $m = \dim(U)$ and *order* $n = \dim(V)$. On homogeneous elements

$$T_G(u \otimes x_1 \otimes \cdots \otimes x_n) = \sum_{\sigma \in G} M(\sigma) u \otimes x_{\sigma^{-1}(1)} \otimes \cdots \otimes x_{\sigma^{-1}(n)}.$$

Remark 1.9 (**Cauchy-Schwartz on symmetric tensors**). Let $e = e_1 \otimes \cdots \otimes e_n$, $x = x_1 \otimes \cdots \otimes x_n$ and $u \in U$, $||u|| = 1$. By the Cauchy-Schwarz inequality,

(1.10) $$|\,(T_G(u \otimes x), T_G(u \otimes e))\,|^2 \leq \|(T_G(u \otimes x)\|^2 \|(T_G(u \otimes e)\|^2.$$

Let $A \in \mathbf{M}_n(\mathbb{C})$, $A = (a_{ij})$, be upper triangular and nonsingular. Define vectors x_i, $i = 1, \ldots, n$, by $x_i = \sum_{j=1}^n a_{ij} e_j$. Thus, $A = (a_{ij}) = ((x_i, e_j))$. We say that A is *row–associated* with the vectors x_i, $i = 1, \ldots, n$ and the basis e_i, $i = 1, \ldots, n$. Marcus ([Mar64]) observed that

$$0 < |\,(T_G(u \otimes x), T_G(u \otimes e))\,|^2 = g^2(\det(A))^2$$
$$\|(T_G(u \otimes x)\|^2 = g(M_{AA^*}u, u) \;\text{ and }\; \|T_G(u \otimes e)\|^2 = g.$$

From definition 1.7 and remarks 1.5, 1.9, we have the following:

Theorem 1.11 (**Cauchy-Schwartz-Schur inequalities).** *Let $u \in U$, $\|u\| = 1$. Let $H \in \mathbf{M}_n(\mathbb{C})$ be positive definite Hermitian. Write $H = AA^*$ where A is upper triangular and nonsingular (e.g., using Cholesky decomposition), and define x_i, $i = 1, \ldots, n$, by $A = ((x_i, e_j))$ (remark 1.9). The following inequalities are equivalent*

- **(1) C-S:** $|\,(T_G(u \otimes x), T_G(u \otimes e))\,|^2 \leq \|T_G(u \otimes x)\|^2 \|T_G(u \otimes e)\|^2$
- **(2) Schur H:** $\det(H) \leq (M_H u, u)$,
- **(3) A form:** $(\det(A))^2 \leq (M_{AA^*}u, u)$.

Proof. This result follows directly from remark 1.9.

□

Referring to theorem 1.11, we have the following:

Theorem 1.12 (**Cauchy-Schwartz-Schur equalities).** *Let $u \in U$ and $H = AA^*$ be as in theorem 1.11. The following equalities are equivalent.*

- (0) **C-S Equality condition:** $T_G(u \otimes x) = kT_G(u \otimes e)$, $k \neq 0$
- (1) **C-S Equality:** $|\,(T_G(u \otimes x), T(u \otimes e))\,|^2 = \|T_G(u \otimes x)\|^2 \|T_G(u \otimes e)\|^2$

(2) **Schur H Equality:** $\det(H) = (M_H u, u)$.

(3) **A Equality:** $(\det(A))^2 = (M_{AA^*} u, u)$

(4) **Schur H Equality Condition:** $\mathbb{G}_H \subseteq G, (M(\sigma)u, u) = \epsilon(\sigma), \sigma \in \mathbb{G}_H$

(5) **A Equality Condition:** $\mathbb{G}_A \subseteq G, (M(\sigma)u, u) = \epsilon(\sigma), \sigma \in \mathbb{G}_A$.

Proof. The equivalence of conditions (0), (1), (2) and (3) follows from theorem 1.11 and the standard condition for equality in the Cauchy-Schwartz inequality (i.e. the equivalence of (0) and (1)).

Condition (5) implies (4) ([Wil69], Lemma 5.3):
Assume (5) and show (4). Note that $H(i,j) \equiv h_{ij} \neq 0$ implies that $H(i,j) = \sum_{k=1}^n A(i,k)A^*(k,j) = \sum_{k=1}^n A(i,k)\bar{A}(j,k) \neq 0$. Thus, there is some s such that $(i,s),(j,s) \in \mathbb{G}_A$ which, by hypothesis, implies that $(M((i,s))u,u) = -1$ and $(M((j,s))u,u) = -1$. Thus, $(i,j) = (i,s)(j,s)(i,s) \in \mathbb{G}_H$ has $(M((i,j))u,u) = -1$, and item (4) holds (we use remark 1.4 (2)). Note this also shows that $G_H \subseteq G_A$.

Condition (4) implies (2) ([Wil69], Lemma 5.2):
Assume (4) and show (2). $M_H = \sum_{\sigma\in G} M(\sigma)\prod_{i=1}^n h_{i\sigma(i)}$, thus $(M_H u,u) = \sum_{\sigma\in G}(M(\sigma)u,u)\prod_{i=1}^n h_{i\sigma(i)}$. Note that $\sigma \in S_n$, $\prod_{i=1}^n h_{i\sigma(i)} \neq 0$ implies $h_{i\sigma(i)} \neq 0$, $1 \le i \le n$, which implies, by definition of $\mathbb{G}_H$, $(i,\sigma(i)) \in \mathbb{G}_H$, $1 \le i \le n$. Recall that σ is a product of transpositions of the form $(i,\sigma(i))$. Each such transposition has $(M((i,\sigma(i)))u,u) = -1$ by our hypothesis. Thus, $\sigma \in \mathbb{G}_H \subseteq G$ and $(M(\sigma)u,u) = \epsilon(\sigma)$. We have shown that

$$(M_H u,u) = \sum_{\sigma\in\mathbb{G}_H}(M(\sigma)u,u)\prod_{i=1}^n h_{i\sigma(i)} = \sum_{\sigma\in S_n}\epsilon(\sigma)\prod_{i=1}^n h_{i\sigma(i)} = \det(H).$$

Finally, item (0) implies (5) is proved in [Wil69] (Theorem 4.2) and below (Theorem 3.30). This completes the proof of theorem 1.12.

□

Remark 1.13 (**Basic observations on theorem 1.12**). In showing (5) implies (4) above, we have shown that $\mathbb{G}_H \subseteq \mathbb{G}_A$. Using the ideas in (4) implies (2), for any matrix $D \in \mathbf{M}_n(\mathbb{C})$,

$$\sum_{\sigma\in G} M(\sigma)\prod_i D(i,\sigma(i)) = \sum_{\sigma\in\mathbb{G}_D\cap G} M(\sigma)\prod_i D(i,\sigma(i)).$$

From theorem 1.12, we will have (4) implies (5). If we take $G = \mathbb{G}_H$ in (4) and $M(\sigma) = \epsilon(\sigma)I_m$, we have $\mathbb{G}_A \subseteq \mathbb{G}_H$ from (5). Reversing this argument using (5) implies (4) we get $\mathbb{G}_H \subseteq \mathbb{G}_A$ (also derived from the argument proving (5) implies (4) above). Thus, $\mathbb{G}_H = \mathbb{G}_A$. A simple example shows the generating sets $\{(i,j) \mid A(i,j) \neq 0\}$ and $\{(i,j) \mid H(i,j) \neq 0\}$ need not be the same. Schur in [Sch18] proved (2) of theorem 1.11 and showed the equivalence of (2) and (4) of theorem 1.12. The focus of this paper is to characterize the multilinear algebraic properties of the symmetry operators T_G that result in the equivalence of (0) and (5) in theorem 1.12. The most interesting structural properties of T_G arise in proving (0) implies (5) (theorem 3.30). The converse (5) implies (0) follows from theorem 1.12.

2. Inequalities

We use the terminology of the previous section. Let U and V be m and n dimensional unitary spaces with the standard inner products.

Remark 2.1 (**Properties of T_G**). To summarize, the $M(\sigma)$ and $P(\sigma)$ are unitary operators: $(M(\sigma))^* = M(\sigma^{-1})$ (by definition of M) and $(P(\sigma))^* = P(\sigma^{-1})$ (simple computation). Thus, $M(\sigma) \otimes P(\sigma)$ is unitary since

$$(M(\sigma) \otimes P(\sigma))^* = (M(\sigma))^* \otimes (P(\sigma))^* = M(\sigma^{-1}) \otimes P(\sigma^{-1}).$$

M is defined to be a representation of G as unitary operators on U, and P is a representation of G as unitary operators on $\otimes^n V$. T_G is essentially idempotent, $T_G^2 = gT_G$, and Hermitian, $T_G^* = T_G$.

Remark 2.2 (**Inner products as products of associated matrices**). Consider the unitary inner product defined by

$$(x_1 \otimes \cdots \otimes x_n, y_1 \otimes \cdots \otimes y_n) = \prod_{i=1}^{n} (x_i, y_i). \tag{2.3}$$

Let $x_i = \sum_{t=1}^n a_{it} e_t$ and $y_j = \sum_{t=1}^n b_{jt} e_t$ where $e_1, \ldots, e_n$ is the orthonormal basis for V. In the unitary space V, we have $(x_i, y_j) = \sum_{t=1}^n a_{it} \bar{b}_{jt}$. In terms of remark 1.9, A is row–associated with the vectors x_i, $i = 1, \ldots, n$ and the basis e_i, $i = 1, \ldots, n$, and B is row–associated with the vectors y_i, $i = 1, \ldots, n$ and the basis e_i, $i = 1, \ldots, n$. With the basis e_i, $i = 1, \ldots, n$ understood, we also refer to A and B as the matrices associated with $x = x_1 \otimes \cdots \otimes x_n$ and $y_1 \otimes \cdots \otimes y_n$ respectively.

For $B = (b_{ij})$, the conjugate transpose $B^*(i,j) = (\bar{b}_{ji})$. Thus, the inner product (x_i, y_i) becomes

$$(x_i, y_j) = \sum_{t=1}^{n} a_{it} \bar{b}_{jt} = \sum_{t=1}^{n} A(i,t) B^*(t,j) = AB^*(i,j). \tag{2.4}$$

If $x = x_1 \otimes \cdots \otimes x_n = y_1 \otimes \cdots \otimes y_n$ then $A = B$ and AA^* is the Gram matrix of the sequence $x_1, \ldots, x_n$ (i.e. the rows of A).

We use the terminology of theorem 1.1 and equation 2.4.

Lemma 2.5 (**Marcus's inner product form for symmetric tensors** [Mar64]). *Let $T_G(u_1 \otimes x)$ and $T_G(u_2 \otimes y)$ be two generalized symmetric tensors where $x = x_1 \otimes \cdots \otimes x_n$, $y = y_1 \otimes \cdots \otimes y_n$, and A and B are the matrices associated with x and y respectively (remark 2.2). Then*

$$(T_G(u_1 \otimes x), T_G(u_2 \otimes y)) = g(M_{AB^*} u_1, u_2). \tag{2.6}$$

Proof. $(T_G(u_1 \otimes x), T_G(u_2 \otimes y)) = (T_G^2(u_1 \otimes x), u_2 \otimes y) = g(T_G(u_1 \otimes x), u_2 \otimes y)$ (for properties of T_G, see remark 2.1).

$$(T_G(u_1 \otimes x), u_2 \otimes y) = \left(\sum_{\sigma \in G} M(\sigma) u_1 \otimes P(\sigma) x, u_2 \otimes y \right) =$$

$$\sum_{\sigma \in G} (M(\sigma) u_1, u_2)(P(\sigma)x, y) = \sum_{\sigma \in G} (M(\sigma) u_1, u_2) \prod_{i=1}^{n} (x_{\sigma^{-1}(i)}, y_i) =$$

$$\sum_{\sigma \in G} (M(\sigma) u_1, u_2) \prod_{i=1}^{n} AB^*(\sigma^{-1}(i), i) \text{ (equation 2.4)}$$

This latter equation becomes

$$\left(\left(\sum_{\sigma \in G} M(\sigma) \prod_{i=1}^{n} AB^*(\sigma^{-1}(i), i) \right) u_1, u_2 \right) = (M_{AB^*} u_1, u_2) .$$

□

Theorem 2.7 (**Marcus's generalization of Schur's inequality** [Mar64]). *Let A and B be $n \times n$ complex matrices. Let M_K be the generalized matrix function of K. Then, for any $u_1, u_2 \in U$,*

$$|(M_{AB^*} u_1, u_2)|^2 \leq (M_{AA^*} u_1, u_1)\,(M_{BB^*} u_2, u_2) . \tag{2.8}$$

Proof. Let $T_G(u_1 \otimes x)$ and $T_G(u_2 \otimes y)$ be generalized symmetric tensors where $x = x_1 \otimes \cdots \otimes x_n$ and $y = y_1 \otimes \cdots \otimes y_n$ are chosen so the the $n \times n$ matrices A and B are associated with x and y respectively (remark 2.2). By the Cauchy-Schwarz inequality, we have

$$|\,(T_G(u_1 \otimes x), T_G(u_2 \otimes y))\,|^2 \leq \tag{2.9}$$

$$(T_G(u_1 \otimes x), T_G(u_1 \otimes x))\;(T_G(u_2 \otimes y), T_G(u_2 \otimes y)) .$$

From lemma 2.5, equation 2.9 becomes

$$g^2 |(M_{AB^*} u_1, u_2)|^2 \leq g\,(M_{AA^*} u_1, u_1)\; g\,(M_{BB^*} u_2, u_2). \tag{2.10}$$

□

Corollary 2.11 (**Schur's inequality from theorem 2.7**). *Let G be a subgroup of the symmetric group S_n of degree n. Let $H = (h_{ij})$ be an $n \times n$ positive definite Hermitian matrix. Let M be a representation of G as unitary linear operators on U, $\dim(U) = m$, and let $M_H = \sum_{\sigma \in G} M(\sigma) \prod_{i=1}^{n} h_{i\sigma(i)}$. Then M_H is a positive definite Hermitian transformation on U and, if $u \in U$ has $||u|| = 1$, then*

$$\det(H) \leq (M_H u, u). \tag{2.12}$$

Proof. Write $M_H = \sum_{\sigma \in G} M(\sigma) \prod_{i=1}^{n} H(i, \sigma(i))$.

$$(M_H)^* = \sum_{\sigma \in G} (M(\sigma))^* \prod_{i=1}^{n} \overline{H(i, \sigma(i))} = \sum_{\sigma \in G} (M(\sigma^{-1})) \prod_{i=1}^{n} H^*(\sigma(i), i) = M_{H^*}.$$

Thus, $H^* = H$ implies that $(M_H)^* = M_H$ and hence M_H is Hermitian. If H is positive definite Hermitian then $\det(H) > 0$, and 2.12 will imply that every eigenvalue of M_H is positive, hence M_H is positive definite Hermitian. To prove 2.12, take $H = AA^*$ and let $B = I_n$ in equation 2.8. In which case $M_{BB^*} = M_{I_n} = I_m$. Assume without loss of generality that A is triangular and take $u = u_1 = u_2$ to be a unit vector. In this case, $(M_A u, u) = \det(A)$ and equation 2.8 becomes $|\det(A)|^2 = \det(H) \leq (M_H u, u)$ which proves 2.12. ∎

3. Equalities

The proof of Schur's inequality, corollary 2.11, was obtained from Marcus's tensor form of the Cauchy-Schwartz inequality

$$\begin{aligned} |\,(T_G(u_1 \otimes x), T_G(u_2 \otimes y))\,|^2 \leq \\ (T_G(u_1 \otimes x), T_G(u_1 \otimes x))\ (T_G(u_2 \otimes y), T_G(u_2 \otimes y)) \end{aligned} \tag{3.1}$$

by taking $u_1 = u_2 = u$ and $y = y_1 \otimes \cdots \otimes y_n = e_1 \otimes \cdots \otimes e_n = e$:

$$|\,(T_G(u \otimes x), T_G(u \otimes e))\,|^2 \leq \|T_G(u \otimes x)\|^2\, |T_G(u \otimes e)\|^2. \tag{3.2}$$

Both $T_G(u \otimes x)$ and $T_G(u \otimes e)$ are nonzero, hence equality occurs in equation 3.2 if and only if

$$T_G(u \otimes x) = kT_G(u \otimes e)\,,\ k \neq 0. \tag{3.3}$$

Remark 3.4 (**The case $T_G(u \otimes x) = kT_G(u \otimes e)\,,\ k \neq 0,\ n = 2$**). Choose $u \in U$, $\|u\| = 1$. The nonsingular upper triangular matrix $A = (a_{ij}) = ((x_i, e_j))$ is in $\mathbf{M}_2(\mathbb{C})$. In this case, $x_1 = a_{11}e_1 + a_{12}e_2$, $x_2 = a_{22}e_2$ and $u \otimes x = u \otimes x_1 \otimes x_2$. From 1.7

$$T_G(u \otimes x) = \sum_{\sigma \in G} M(\sigma) u \otimes x_{\sigma^{-1}(1)} \otimes x_{\sigma^{-1}(2)}.$$

We have assumed that

$$T_G(u \otimes x) = kT_G(u \otimes e)\,,\ k \neq 0. \tag{3.5}$$

If $G = \{\iota, \tau\}$, $\tau = (1, 2)$, we compute directly that

(3.6)

$$T_G(u \otimes x) = a_{11}a_{22}\,(u \otimes e_1 \otimes e_2 + M(\tau)u \otimes e_2 \otimes e_1) + a_{12}a_{22}Su \otimes e_2 \otimes e_2$$

where $S = I_m + M(\tau)$. For $x = e$ (for which $A = I_2$) we get

$$T_G(u \otimes e) = u \otimes e_1 \otimes e_2 + M(\tau)u \otimes e_2 \otimes e_1. \tag{3.7}$$

Let $\mathbb{G}_A$ be as in remark 1.4(3). From equations 3.6 and 3.7 we see that equation 3.5 holds (for $G = \{\iota, \tau\}$) if and only if either

$$(1)\, \mathbb{G}_A = \{\iota\}\ \ (a_{12} = 0)\ \text{ or }\ (2)\, \mathbb{G}_A = \{\iota, \tau\}\ \ (a_{12} \neq 0)\ \text{ and }\ Su = 0.$$

SInce $S = I_m + M(\tau)$, we have $(Su, u) = 0$ if and only if $(M(\tau)u, u) = -1$. In the trivial case where $G = \{\iota\}$, equation 3.5 holds if and only if $\mathbb{G}_A = \{\iota\}$ (i.e., $a_{12} = 0$):

$$T_G(u \otimes x) = a_{11}a_{22}u \otimes e_1 \otimes e_2 + a_{12}a_{22}u \otimes e_2 \otimes e_2 \tag{3.8}$$

and

$$T_G(u \otimes e) = u \otimes e_1 \otimes e_2. \tag{3.9}$$

To summarize, in the $n = 2$ case, equation 3.5 holds if and only if $\mathbb{G}_A$ is contained in G and $(M(\sigma)u, u) = \epsilon(\sigma)$ for $\sigma \in \mathbb{G}_A$.

Definition 3.10 (Compatible permutations). Let $\alpha \in \Gamma_n$. A permutation $\sigma \in S_n$ will be α-compatible if $\alpha(\sigma(i)) \geq i$ for all $i \in \underline{n}$. Let

$$S_n^\alpha = \{\sigma \mid \sigma \in S_n, \alpha(\sigma(i)) \geq i, i \in \underline{n}\}$$

denote the set of all α-compatible permutations.

Definition 3.11 (Restricted α-compatible permutations). $T_G(u \otimes x)$ has upper triangular matrix $A = (a_{ij}) = ((x_i, e_j))$ associated with $x = x_1 \otimes \cdots \otimes x_n$ and the basis $\{e_i \mid i = 1, \ldots n\}$. Define

$$S_n^\alpha(A) \equiv S_n^\alpha(x, e) = \{\sigma \mid \sigma \in S_n^\alpha, \prod_{i=1}^{n} a_{i\alpha\sigma(i)} \neq 0\}. \tag{3.12}$$

We call $S_n^\alpha(A) \equiv S_n^\alpha(x, e)$ the α-compatible permutations restricted by A or, alternatively, the α-compatible permutations restricted by $x = x_1 \otimes \cdots \otimes x_n$ and the orthonormal basis e_i, $1 \leq i \leq n$.

Remark 3.13 (**Inner products and α-compatible permutations**). Let $u \in U$, $\|u\| = 1$. Let $T_G(u \otimes x)$ be as in definition 3.11. Then

$$(T_G(u \otimes x), u \otimes e_\alpha) = \sum_{\sigma \in G} (M(\sigma)u, u) \prod_{i=1}^{n} \left(x_i, e_{\alpha\sigma(i)}\right). \tag{3.14}$$

Note that A upper triangular implies that $\prod_{i=1}^{n} \left(x_i, e_{\alpha\sigma(i)}\right) = 0$ if $\sigma \notin S_n^\alpha$ (definition 3.10). In fact, $\prod_{i=1}^{n} \left(x_i, e_{\alpha\sigma(i)}\right) = 0$ if $\sigma \notin S_n^\alpha(x, e) \subseteq S_n^\alpha$ (definition 3.11) and we have

$$(T_G(u \otimes x), u \otimes e_\alpha) = \sum_{\sigma \in G \cap S_n^\alpha(x,e)} (M(\sigma)u, u) \prod_{i=1}^{n} \left(x_i, e_{\alpha\sigma(i)}\right). \tag{3.15}$$

We now discuss the general case of equality as it relates to collinearity:

$$T_G(u \otimes x) = kT_G(u \otimes e)\,,\ k \neq 0 \tag{3.16}$$

The proofs will follow the ideas and notation developed in remark 3.4.

Remark 3.17 (**Terminology and notation**). If $K \subseteq \underline{n}$ then $S_K = \mathrm{PER}(K)$ denotes all permutations of K. Thus, $S_n = S_{\underline{n}}$. The set of all functions from K to K is denoted by Γ_K. Thus, $\Gamma_n = \Gamma_{\underline{n}}$. For $p \in \underline{n}$.

$$\Gamma_{n,p} = \{\gamma \mid \gamma \in \Gamma_n, \gamma(p) = p, \gamma(i) \neq p \text{ if } i \neq p\}.$$

Let $G_p = \{\sigma \mid \sigma(p) = p\}$ denote the stability subgroup of G at p. Let $K_p = \underline{n} \setminus \{p\} = \{1, \ldots, p-1\} \cup \{p+1, \ldots n\}$. Let $G'_p = G_p|K_p$ be the restriction of the stability subgroup G_p to K_p. For $\gamma \in \Gamma_{n,p}$ let $\gamma'_p = \gamma|K_p$ and note that $\Gamma_{K_p} = \{\gamma'_p \mid \gamma \in \Gamma_{n,p}\}$. Also, $|\Gamma_{n,p}| = |\Gamma_{K_p}| = (n-1)^{n-1}$, the map $\gamma \mapsto \gamma'_p$ providing the canonical bijection.

If the tensor $x = x_1 \otimes \ldots \otimes x_n$ has associated matrix $A = (a_{ij}) = ((x_i, e_j))$, then define the tensor x^p and vectors x_i^p, $i \in K_p$, to have associated matrix $A(p \mid p)$ in the same sense. Here we use the standard notation which defines $A(p \mid p)$ to be the submatrix of A gotten by deleting row p and column p from A. Thus,

$$x^p = x_1^p \otimes \cdots \otimes x_{p-1}^p \otimes x_{p+1}^p \otimes \cdots \otimes x_n^p.$$

Let $T_G = \sum_{\sigma \in G} M(\sigma) \otimes P(\sigma)$ be a generalized symmetry operator on $U \otimes [\otimes^n V]$ where $\dim(V) = n$ and $\dim(U) = m$. Note that T_G has *degree* m and *order* n (definition 1.7). V has orthonormal basis $\{e_i \mid 1 \leq i \leq n\}$ and $\otimes^n V$ has orthonormal basis $\{e_\alpha \mid \alpha \in \Gamma_n\}$ where $e_\alpha = e_{\alpha(1)} \otimes \cdots \otimes e_{\alpha(n)}$. Let $T_{G'_p} = \sum_{\sigma \in G'_p} M(\sigma) \otimes P(\sigma)$ act on $U \otimes [\otimes^{n-1} V'_p]$. V'_p has orthonormal basis $\{e_i \mid i \in K_p\}$ and $\otimes^{n-1} V'_p$ has orthonormal basis $\{e_{\gamma'_p} \mid \gamma'_p \in \Gamma_{K_p}\}$ where $e_{\gamma'_p} = \otimes_{i \in K_p} e_{\gamma'_p(i)}$. $T_{G'_p}$ has degree m and order $n-1$.

Remark 3.18 (**Maximal row spike functions**). In the following discussion we will use the notions of *spike functions* (definition 4.1), *maximal row spike functions*, α_{rc} (definition 4.3), and the characterization of permutations compatible with maximal row spike functions (lemma 4.4). The statements of these two definitions and the lemma are all that will be needed to understand the proofs that follow. In fact, lemma 4.4 can easily be proved directly from these two definitions. This material has been included in a separate section in order to state and prove lemma 4.2 which is the natural combinatorial setting for this discussion.

Lemma 3.19 (**If columns $A^{(c)}$, $c > 1$, have $r < c$ with $a_{rc} \neq 0$**). *Choose $u \in U$, $\|u\| = 1$. Suppose, for some $k \neq 0$, $T_G(u \otimes x) = kT_G(u \otimes e)$, and, for all $1 < c \leq n$, column $A^{(c)}$ has some $r < c$ with $a_{rc} \neq 0$. Then $\mathbb{G}_A = G = S_n$ and $(M(\sigma)u, u) = \epsilon(\sigma)$ for all σ.*

Proof. The condition on the columns $A^{(c)}$ implies that every column ($c > 1$) has a *maximum row spike function* α_{rc} (definition 4.3). Consider the equation

$$(T_G(u \otimes x), u \otimes e_{\alpha_{rc}}) = k\,(T_G(u \otimes e), u \otimes e_{\alpha_{rc}})\,. \tag{3.20}$$

By equation 3.15,

(3.21)

$$(T_G(u \otimes x), u \otimes e_{\alpha_{rc}}) = \sum_{\sigma \in G \cap S_n^{\alpha_{rc}}(x,e)} (M(\sigma)u, u) \prod_{i=1}^{n} \left(x_i, e_{\alpha_{rc}\sigma(i)}\right).$$

By lemma 4.4, $S_n^{\alpha_{rc}}(x, e) = \{\iota, \tau\}$ where ι is the identity permutation and $\tau = (r, c)$ is a transposition. Thus, $(T_G(u \otimes x), u \otimes e_{\alpha_{rc}}) =$

$$(I_m u, u) \left(\prod_{i \neq r} (x_i, e_i) \right) (x_r, e_c) \ \text{ if } \ \tau \notin G \ \text{ and} \tag{3.22}$$

$$[(I_m u, u) + (M(\tau)u, u)] \left(\prod_{i \neq r} (x_i, e_i) \right) (x_r, e_c) \ \text{ if } \ \tau \in G. \tag{3.23}$$

Note that by the definition of the maximal row spike function, α_{rc}, we have $(x_r, e_c) = a_{rc} \neq 0$. However, $(T_G(u \otimes e), u \otimes e_{\alpha_{rc}}) = 0$. Thus, equation 3.23, not equation 3.22, must hold, and $[(u, u) + (M(\tau)u, u)] = 0$ or $(M(\tau)u, u) = \epsilon(\sigma)$. By hypothesis, for every column, $A^{(c)}$, $c > 1$, there is a maximum row spike function α_{rc}. Thus, we have shown that for every c, $1 < c \leq n$, there is an $r < c$ such that $\tau = (r, c) \in \mathbb{G}_A$ satisfies $[(u, u) + (M(\tau)u, u)] = 0$ or $(M(\tau)u, u) = \epsilon(\sigma)$. This set of transpositions, τ, generates S_n (a trivial induction). We have proved that $(M(\tau)u, u) = \epsilon(\sigma)$ for all $\sigma \in \mathbb{G}_A = S_n = G$ (note remark 1.4 here). ∎

Lemma 3.24 (**General case where column $A^{(p)}$ has only $a_{pp} \neq 0$**). *Choose $u \in U$, $\|u\| = 1$. Assume for $k \neq 0$, $T_G(u \otimes x) = kT_G(u \otimes e)$. Using the terminology of remark 3.17, assume that x is associated with the upper triangular and nonsingular matrix $A = (a_{ij})$. Assume for some p, $1 < p \leq n$, the only nonzero entry in column $A^{(p)}$ is the diagonal entry a_{pp}. Then, for any $\gamma \in \Gamma_{n,p}$,*

$$(T_G(u \otimes x), u \otimes e_\gamma) = a_{pp} \left(T_{G'_p}(u \otimes x^p), u \otimes e_{\gamma'_p} \right). \tag{3.25}$$

Proof. Consider $u \otimes e_\gamma$, $\gamma \in \Gamma_{n,p}$. Then,

$$(T_G(u \otimes x), u \otimes e_\gamma) = \sum_{\sigma \in G} (M(\sigma)u, u) \prod_{i=1}^{n} (x_{\sigma^{-1}(i)}, e_{\gamma(i)}) \tag{3.26}$$

where

$$\prod_{i=1}^{n} (x_{\sigma^{-1}(i)}, e_{\gamma(i)}) = (x_{\sigma^{-1}(1)}, e_{\gamma(1)}) \cdots (x_{\sigma^{-1}(p)}, e_{\gamma(p)}) \cdots (x_{\sigma^{-1}(n)}, e_{\gamma(n)}).$$

Note that $(x_{\sigma^{-1}(p)}, e_{\gamma(p)}) = (x_{\sigma^{-1}(p)}, e_p) \neq 0$ requires $\sigma^{-1}(p) = p$. Thus, $(x_{\sigma^{-1}(p)}, e_p) = (x_p, e_p) = a_{pp} \neq 0$. Hence the sum in equation 3.26 can be taken over G_p, the stabilizer subgroup of G at p. We have

$$(T_G(u \otimes x), u \otimes e_\gamma) = a_{pp} \sum_{\sigma \in G_p} (M(\sigma)u, u) \prod_{i \neq p} (x_{\sigma^{-1}(i)}, e_{\gamma(i)}). \tag{3.27}$$

As in remark 3.17, let G'_p be G_p restricted to K_p, and let $x^p = x_1^p \otimes \ldots x_{p-1}^p \otimes x_{p+1}^p \otimes x_n^p$ be the tensor that has matrix $A(p \mid p)$ with respect to the orthonormal basis $\{e_i \mid i \in K_p\}$. Let γ'_p denote γ restricted to K_p.

The unitary representation M of G restricts in the obvious way to G_p and ${G'}_p$. Let $V'_p = \langle e_i \mid i \in K_p \rangle$. The generalized symmetry operator $T_{G'_p} = \sum_{\sigma \in G'_p} M(\sigma) \otimes P(\sigma)$ acting on $U \otimes [\otimes V'_p]$ allows us to reformulate equation 3.27 as follows:

$$(T_G(u \otimes x), u \otimes e_\gamma) = a_{pp} \left(T_{G'_p}(u \otimes x^p), u \otimes e_{\gamma'_p} \right). \tag{3.28}$$

□

Remark 3.29 (**Inductive step using equation 3.25**). From the terminology of remark 3.17 we note that $T_{G'_p} = \sum_{\sigma \in G'_p} M(\sigma) \otimes P(\sigma)$ acts on $U \otimes [\otimes^{n-1} V'_p]$. V'_p has orthonormal basis $\{e_i \mid i \in K_p\}$ and $\otimes^{n-1} V'_p$ has orthonormal basis $\{e_{\gamma'_p} \mid \gamma'_p \in \Gamma_{K_p}\}$ where $e_{\gamma'_p} = \otimes_{i \in K_p} e_{\gamma'_p(i)}$. Thus, $T_{G'_p}$ has degree m and order $n-1$ and will be used in the inductive step (on n) in the next theorem.

Theorem 3.30. *Let $u \in U$, $\|u\| = 1$. Let $A = (a_{ij}) = ((x_i, e_j))$ be the upper triangular nonsingular matrix associated with $x = x_1 \otimes \cdots \otimes x_n$ and basis e_i, $1 \leq i \leq n$. If $T_G(u \otimes x) = kT_G(u \otimes e)$, $k \neq 0$, then $\mathbb{G}_A$ is a subgroup of G and $(M(\sigma)u, u) = \epsilon(\sigma)$ for all $\sigma \in \mathbb{G}_A$.*

Proof. If for all c, $1 < c \leq n$, column $A^{(c)}$ has some $r < c$ with $a_{rc} \neq 0$ then the result follows from lemma 3.19. Otherwise, for some p, $1 < p \leq n$, the only nonzero entry in column $A^{(p)}$ is the diagonal entry a_{pp}. We apply lemma 3.24, in particular

$$(T_G(u \otimes x), u \otimes e_\gamma) = a_{pp} \left(T_{G'_p}(u \otimes x^p), u \otimes e_{\gamma'_p} \right) \tag{3.31}$$

and for $x = e$:

$$(T_G(u \otimes e), u \otimes e_\gamma) = \left(T_{G'_p}(u \otimes e^p), u \otimes e_{\gamma'_p} \right). \tag{3.32}$$

The proof is by induction on the order n of T_G (i.e., $n = \dim(V)$). The base case, $n = 2$, is established in remark 3.4. Assume the theorem is true for generalized symmetry operators of degree m and order $n - 1$. Note that $T_G(u \otimes x) = kT_G(u \otimes e)$, $k \neq 0$, implies that $(T_G(u \otimes x), u \otimes e_\gamma) = k(T_G(u \otimes e), u \otimes e_\gamma)$ for all $\gamma \in \Gamma_n$. Thus, by equations 3.31 and 3.32

$$a_{pp}(T_{G'_p}(u \otimes x^p), u \otimes e^p_{\gamma'_p}) = k(T_{G'_p}(u \otimes e^p), u \otimes e^p_{\gamma'_p}) \tag{3.33}$$

for all $\gamma'_p \in \Gamma_{K_p}$. But, $\{u \otimes e_{\gamma'_p} \mid \gamma'_p \in \Gamma_{K_p}\}$ spans $U \otimes [\otimes^{n-1} V_p]$ (3.17). Thus,

$$(T_{G'_p}(u \otimes x^p)) = k_p(T_{G'_p}(u \otimes e^p)), \quad k_p = k/a_{pp}. \tag{3.34}$$

$T_{G'_p}$ has degree m and order $n - 1$. Likewise,

$$T_{G'_1}(u \otimes x^1) = k_1 T_{G'_1}(u \otimes e^1), \quad k_1 = k/a_{11}. \tag{3.35}$$

$T_{G'_1}$ has degree m and order $n - 1$. The tensor x^1 is associated with the upper triangular nonsingular submatrix $B = A(1|1)$. By 3.35 and the induction hypothesis, $\mathbb{G}_B$ is a subgroup of G'_1, the stabilizer subgroup of G at 1 restricted to K_1, and $(M(\sigma)u, u) = \epsilon(\sigma)$ for $\sigma \in G_B$. Let $C = A(p|p)$. C is also upper triangular and nonsingular. By 3.34 and the induction hypothesis, $\mathbb{G}_C$ is a subgroup of G'_p, the stabilizer subgroup of G at p restricted to K_p, and $(M(\sigma)u, u) = \epsilon(\sigma)$ for $\sigma \in \mathbb{G}_C$. Observe that the union of the set of generating transpositions $\{(i,j) \mid i < j,\, B(i,j) \neq 0\}$, together with the set $\{(i,j) \mid i < j,\, C(i,j) \neq 0\}$, equals the set $\{(i,j) \mid i < j,\, A(i,j) \neq 0\}$. Here we use the fact that $a_{1p} = 0$. Thus, $\mathbb{G}_A$ is a subgroup of G and $(M(\sigma)u, u) = \epsilon(\sigma)$ for all $\sigma \in G_A$. We use remark 1.4(2) here. ∎

4. Remarks about compatible permutations

We combine here some interesting combinatorial lemmas about compatible permutations. We have used only lemma 4.4 which can be proved independently as an exercise.

Definition 4.1 (Spike functions). Let $r, c \in \underline{n}$, $r < c$. The r, c *spike function*, α_{rc}, is defined by

$$\alpha_{rc}(r) = c \text{ and } \alpha_{rc}(i) = i,\ i \neq r.$$

The following lemma characterizes the spike function compatible permutations ([Wil69], Lemma 3.3, p. 339).

Lemma 4.2 (Spike function compatible permutations). *Let α_{rc} be a spike function (4.1). The set of all α_{rc}-compatable permutations, $S_n^{\alpha_{rc}}$, consists of the identity permutation, ι, together with all σ which have only one cycle of length greater than one. Moreover, that cycle can be written in the form $(r_1, \ldots, r_p)$ where $r = r_1 < \cdots < r_p \leq c$. Thus, $|S_n^{\alpha_{rc}}| = 2^{c-r}$.*

Proof. Let $(r_1, \ldots, r_p)$ be a nontrivial cycle of σ. Assume without loss of generality that $r_1 = \min\{r_1, \ldots, r_p\}$. Suppose $r_1 \in \{1, \ldots, r-1, r+1, \ldots n\}$ (i.e., $r_1 \neq r$). Then $\alpha_{rc}(\sigma(r_p)) = \alpha_{rc}(r_1) = r_1 < r_p$ which contradicts the assumption $\sigma \in S_n^{\alpha_{rc}}$. Thus, $r_1 = r$ which implies there is only one nontrivial cycle of σ. If $r_p > c$, then $\alpha_{rc}(\sigma(r_p)) = \alpha_{rc}(r) = c < r_p$, again contradicting $\sigma \in S_n^{\alpha_{rc}}$. Thus, $r_p \leq c$ and we need only show the strictly increasing property: $r = r_1 < \cdots < r_p$. By minimality of r_1, we have $r_1 < r_2$. Suppose we have $r_1 < r_2 < \cdots < r_{t-1} > r_t$ for some $2 < t \leq p$. Then $\alpha_{rc}(\sigma(r_{t-1})) = \alpha_{rc}(r_t) = r_t < r_{t-1}$ which contradicts the assumption $\sigma \in S_n^{\alpha_{rc}}$. Thus the unique nontrivial cycle of σ is of the form $(r_1, \ldots, r_p)$ which we may assume satisfies $r = r_1 < \cdots < r_p \leq c$. These cycles can be constructed by choosing a nonempty subset of $\{r+1, \ldots, c\}$. This can be done in $2^{c-r} - 1$ ways. Counting the identity permutation, this gives $|S_n^{\alpha_{rc}}| = 2^{c-r}$.

□

Definition 4.3 (Maximal row spike functions). We use the terminology of remark 3.13. Let $A = (a_{ij}) = ((x_i, e_j))$ be the upper triangular nonsingular matrix associated with $x_1 \otimes \cdots \otimes x_n$ and basis $e_1, \ldots, e_n$. Suppose column $A^{(c)}$ has at least two nonzero entries. Let $r = \max\{i \mid i < c, a_{ic} \neq 0\}$. We call the spike function α_{rc} the *maximal row* spike function for column c. If $\{i \mid i < c, a_{ic} \neq 0\}$ is empty, we say column c has no maximal row spike function.

We next characterize the sets $S_n^\alpha(x,e)$ for maximal row spike functions $\alpha = \alpha_{rc}$.

Lemma 4.4 (**Permutations compatible with maximal row spike functions).** *We use definition 4.3 and the terminology of remark 3.13. Let $A = (a_{ij}) = ((x_i, e_j))$ be the upper triangular matrix associated with $x = x_1 \otimes \cdots \otimes x_n$ and the orthonormal basis $\{e_i \mid i = 1, \ldots n\}$. Let α_{rc} be a maximal row spike function for column $A^{(c)}$. Then*

$$S_n^{\alpha_{rc}}(x,e) = \{\iota, \tau\} \tag{4.5}$$

where ι is the identity and $\tau = (r,c)$.

Proof. Note that $S_n^\alpha(x,e) \subseteq S_n^{\alpha_{rc}}$. By lemma 4.2, $\sigma \in S_n^{\alpha_{rc}}$ implies $\sigma = (r_1, \ldots, r_p)$ where $r = r_1 < \cdots < r_p \leq c$. If $r_p < c$ then

$$A(r_p, \alpha_{rc}(\sigma(r_p)) = A(r_p, \alpha_{rc}(r)) = A(r_p, c) = 0$$

by maximality of r. Thus, by equation 3.12, $\sigma \notin S_n^{\alpha_{rc}}(x,e)$. Thus, $r_p = c$. If $p > 2$, we have $1 < r_{p-1} < c = r_p$.

$$A(r_{p-1}, \alpha_{rc}(\sigma(r_{p-1})) = A(r_{p-1}, \alpha_{rc}(c)) = A(r_{p-1}, c) = 0$$

by the maximality of r. Again by equation 3.12, $\sigma \notin S_n^{\alpha_{rc}}(x,e)$. Thus, $p = 2$. We have shown that $S_n^{\alpha_{rc}}(x,e) = \{\iota, \tau\}$ where ι is the identity and $\tau = (r,c)$.

□

5. Character form and examples of Schur's Theorem

The following definition and terminology will be useful:

Definition 5.1 (**Schur generalized matrix function).** Let G be a subgroup of the symmetric group S_n of degree n. Let $K = (k_{ij})$ (alternatively, $k_{ij} \equiv K(i,j)$) be an $n \times n$ matrix with complex entries. Let M be a representation of G as unitary linear operators on a unitary space U, $\dim U = m$. Then the *Schur generalized matrix function*, M_K, is defined as follows:

$$M_K = \sum_{\sigma \in G} M(\sigma) \prod_{i=1}^{n} K(\sigma^{-1}(i), i) = \sum_{\sigma \in G} M(\sigma) \prod_{i=1}^{n} K(i, \sigma(i)). \tag{5.2}$$

Remark 5.3 (**Character or trace form of Schur's theorem**). We use the notation of theorem 1.1. M_H is positive definite Hermitian. Let

$\mathcal{E}_{M_H}$ denote the multiset of eigenvalues of M_H (i.e., eigenvalues with multiplicities). We have $\lambda > 0$ for all $\lambda \in \mathcal{E}_{M_H}$ and

$$\sum_{\lambda \in \mathcal{E}_{M_H}} \lambda = \mathrm{Tr}(M_H) = \sum_{\sigma \in G} \mathrm{Tr}(M(\sigma)) \prod_{i=1}^{n} h_{i\sigma(i)}. \tag{5.4}$$

Let $u_{\min}$ be a unit eigenvector corresponding to $\lambda_{\min}$, the minimum eigenvalue in $\mathcal{E}_{M_H}$. From theorem 1.1 we have $\det(H) \le (M_H u_{\min}, u_{\min}) = \lambda_{\min}$. From equation 5.4, $m \det(H) \le m\lambda_{\min} \le \mathrm{Tr}(M_H)$. Thus, $m\det(H) = \mathrm{Tr}(M_H)$ if and only if $\mathcal{E}_{M_H} = \{\det(H), \ldots, \det(H)\}$ (i.e., $\det(H)$ has multiplicity m). In this case, $M_H = \det(H) I_m$, I_m the identity. We have shown

$$m \det(H) \le \mathrm{Tr}(M_H) \text{ and} \tag{5.5}$$
$$m \det(H) = \mathrm{Tr}(M_H) \iff M_H = \det(H) I_m.$$

As an example, take $G = S_3$, the symmetric group on $\{1, 2, 3\}$, and take

$$M(\sigma) = \begin{pmatrix} \epsilon(\sigma) & 0 \\ 0 & 1 \end{pmatrix}.$$

Let H be a 3×3 positive definite Hermitian matrix such that the permanent, $\mathrm{per}(H) > \det(H)$. Then

$$M_H = \begin{pmatrix} \det(H) & 0 \\ 0 & \mathrm{per}(H) \end{pmatrix}.$$

In this case,

$$(M_H u_{\min}, u_{\min}) = \lambda_{\min} = \det(H).$$

Note that the equality condition of 5.5 does not hold: $2\det(H) < \mathrm{Tr}(M_H) = \det(H) + \mathrm{per}(H)$. To compare this with the equality condition 1.3, note that $\mathbb{G}_H \subseteq S_3$ holds trivially. Let $u = (u_1, u_2) \in U$ have $\|u\| = 1$. Condition 1.3 states that $(M_H u, u) = \det(H)$ if and only if $(M(\sigma)u, u) = \epsilon(\sigma)$ for all $\sigma \in \mathbb{G}_H$. In this case these conditions hold if and only if $u = (e^{ir}, 0)$. Note that $\mathrm{per}(H) > \det(H)$ implies H has at least one off diagonal element.

Remark 5.6 (**Examples of Schur's theorem**). We consider an example of $M_H = \sum_{\sigma \in G} M(\sigma) \prod_{i=1}^{n} h_{i\sigma(i)}$. Take $G = S_3$, the symmetric group on $\{1, 2, 3\}$. Take the unitary representation to be the following:

$$M(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad M((123)) = \begin{pmatrix} \frac{-1}{2} & \frac{+\sqrt{3}}{2} \\ \frac{-\sqrt{3}}{2} & \frac{-1}{2} \end{pmatrix} \quad M((132)) = \begin{pmatrix} \frac{-1}{2} & \frac{-\sqrt{3}}{2} \\ \frac{+\sqrt{3}}{2} & \frac{-1}{2} \end{pmatrix}$$

$$M(23) = \begin{pmatrix} \frac{-1}{2} & \frac{+\sqrt{3}}{2} \\ \frac{+\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \quad M((13)) = \begin{pmatrix} \frac{-1}{2} & \frac{-\sqrt{3}}{2} \\ \frac{-\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \quad M((12)) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The Schur generalized matrix function, M_H is

$$M_H = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} h_{11}h_{22}h_{33} + \begin{pmatrix} \frac{-1}{2} & \frac{+\sqrt{3}}{2} \\ \frac{-\sqrt{3}}{2} & \frac{-1}{2} \end{pmatrix} h_{12}h_{23}h_{31} + \begin{pmatrix} \frac{-1}{2} & \frac{-\sqrt{3}}{2} \\ \frac{+\sqrt{3}}{2} & \frac{-1}{2} \end{pmatrix} h_{13}h_{21}h_{32} +$$

$$\begin{pmatrix} \frac{-1}{2} & \frac{+\sqrt{3}}{2} \\ \frac{+\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} h_{11}h_{23}h_{32} + \begin{pmatrix} \frac{-1}{2} & \frac{-\sqrt{3}}{2} \\ \frac{-\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} h_{13}h_{22}h_{31} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} h_{12}h_{21}h_{33}.$$

Let

$$H = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & i \\ 0 & -i & 1 \end{pmatrix} \quad \text{which gives} \quad M_H = \begin{pmatrix} \frac{5}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{7}{2} \end{pmatrix}$$

The eigenvalues $\mathcal{E}_{M_H} = \{2, 4\}$, $\det(H) = 2 < \mathrm{Trace}(M_H) = 6$. Also,

$$(M_H u_{\min}, u_{\min}) = \lambda_{\min} = 2 = \det(H)$$

By theorem 1.1, equation 1.3, $\det(H) = (M_H u_{\min}, u_{\min})$ implies that

$$\mathbb{G}_H \subseteq G \text{ and } (M(\sigma)u_{\min}, u_{\min}) = \epsilon(\sigma),\ \sigma \in \mathbb{G}_H.$$

By definition (in theorem 1.1), $\mathbb{G}_H$ is the group generated by $\{e, (2,3)\}$. We have, trivially, $(M(e)u_{\min}, u_{\min}) = 1$. To compute $(M(2,3)u_{\min}, u_{\min})$ we evaluate the minimum eigenvector $u_{\min}$ for M_H and get $u_{\min} = (\frac{-\sqrt{3}}{2}, \frac{1}{2})$ and compute $M((23))u_{\min}$ where

$$M((23)) = \begin{pmatrix} \frac{-1}{2} & \frac{+\sqrt{3}}{2} \\ \frac{+\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$$

getting $-u_{\min}$ so $(M((23))u_{\min}, u_{\min}) = -1 = \epsilon((23))$ as required by the equality condition 1.3.

Acknowledgment: The author wishes to thank Dr. Tony Trojanowski for many helpful observations and specific suggestions that improved this paper.

References

[Mar64] Marvin Marcus. On two classical results of I. Schur. *Bull. Amer. Math. Soc.*, 70:685–688, 1964.

[Sch18] Isaiah Schur. Uber endlich Gruppen und Hermitesche Formen. *Math Z.*, 1:184–207, 1918.

[Wil69] S. G. Williamson. Tensor Contraction and Hermitian Forms. *Linear Algebra and Its Applications*, 2:335–347, 1969.