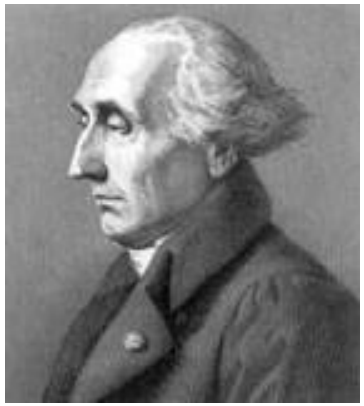
Algorithms for the Densest Sublattice Problem

Daniele Micciancio (UCSD)
(Joint work with D. Dadush – SODA 2013)

January 2013

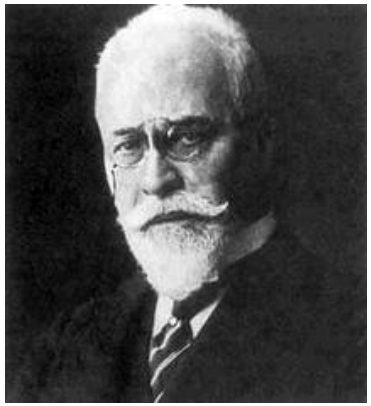
(Point) Lattices

- Traditional area of mathematics



Lagrange

Gauss



Minkowski

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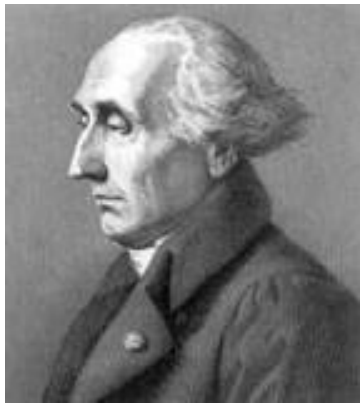

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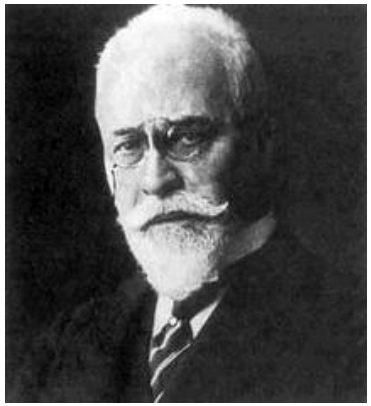
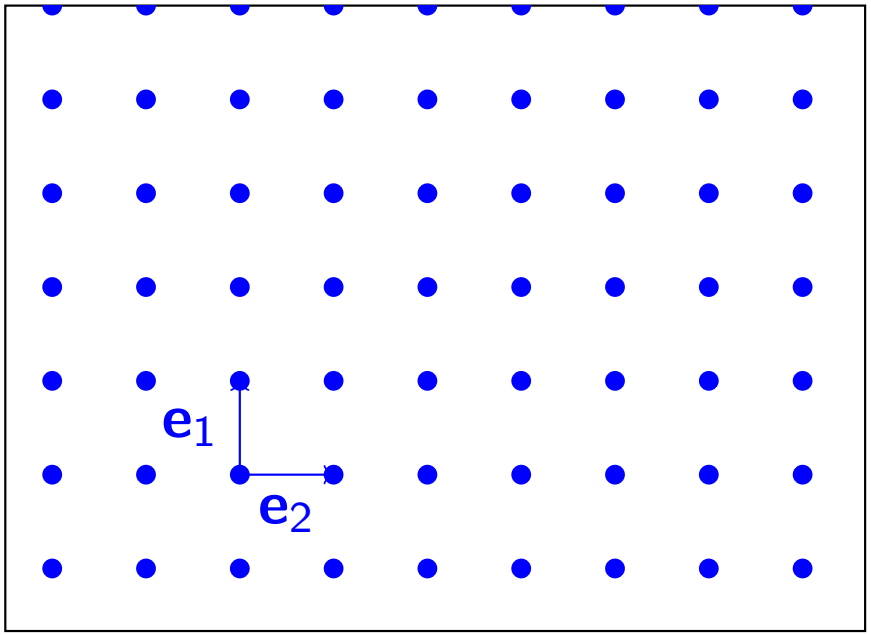


Minkowski

- Key to many algorithmic applications
  - Cryptanalysis (e.g., breaking low-exponent RSA)
  - Coding Theory (e.g., wireless communications)
  - Optimization (e.g., Integer Programming with fixed number of variables)
  - Cryptography (e.g., Cryptographic functions from worst-case complexity assumptions, Fully Homomorphic Encryption)

Outline

Lattices: Definition



The simplest lattice in n-dimensional space is the integer lattice

$$\Lambda = \mathbb{Z}^n$$

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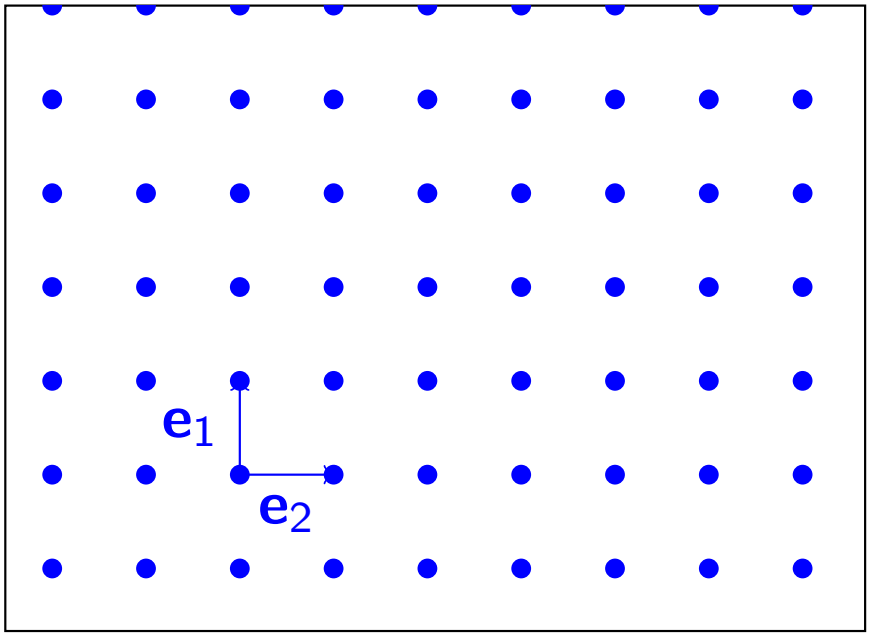

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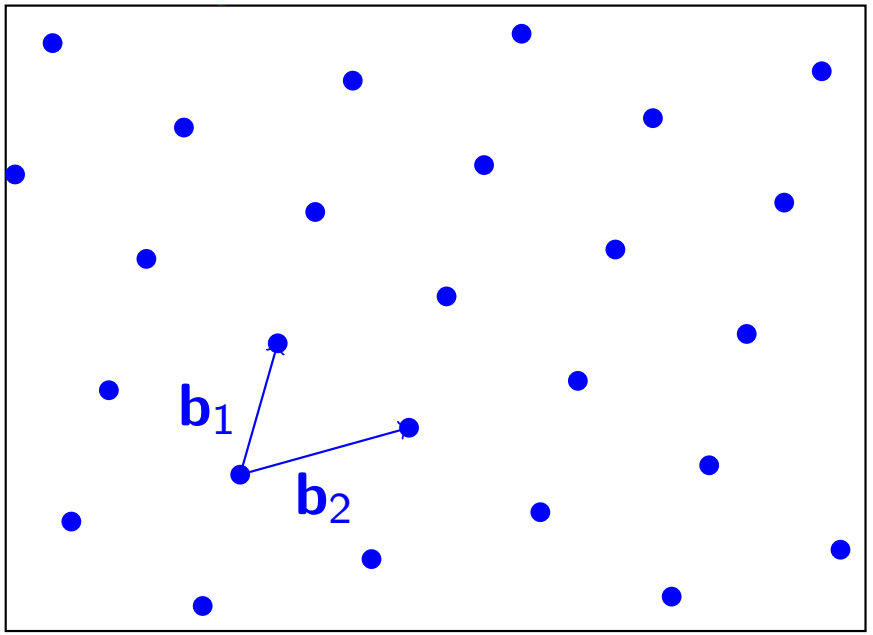
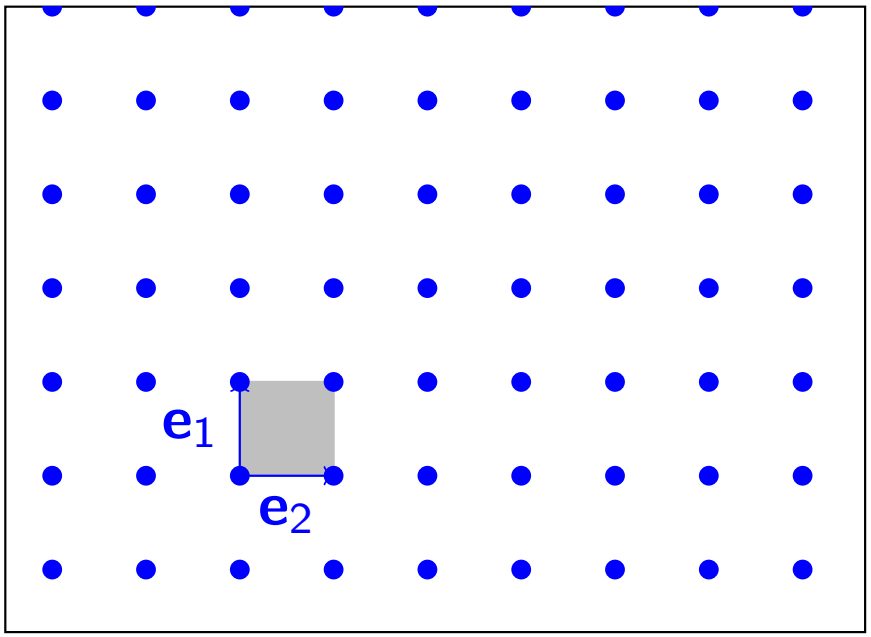
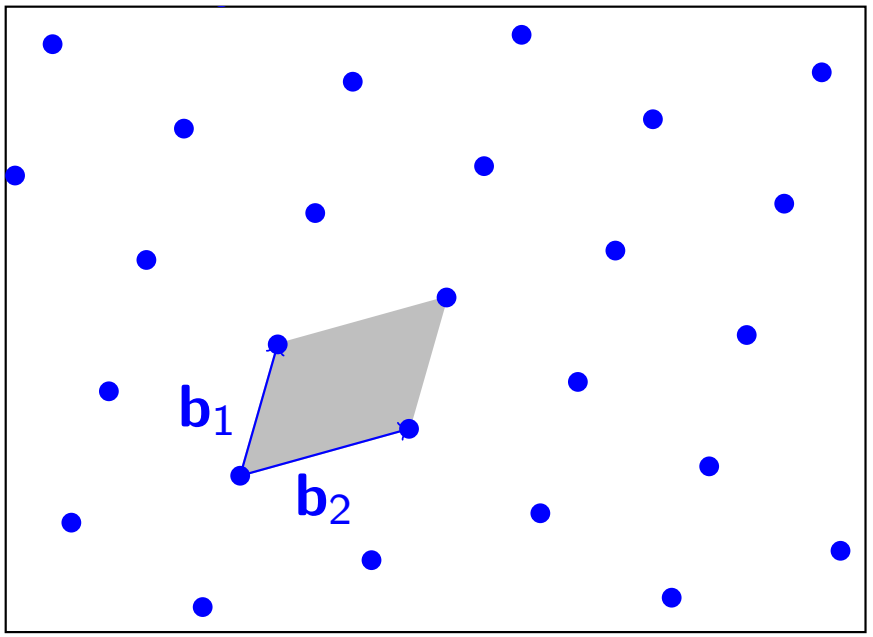


Other lattices are obtained by applying a linear transformation

$$\Lambda = \mathbf{B}\mathbb{Z}^n \qquad (\mathbf{B} \in \mathbb{R}^{d \times n})$$

Lattice Determinant / Density





Definition (Determinant)

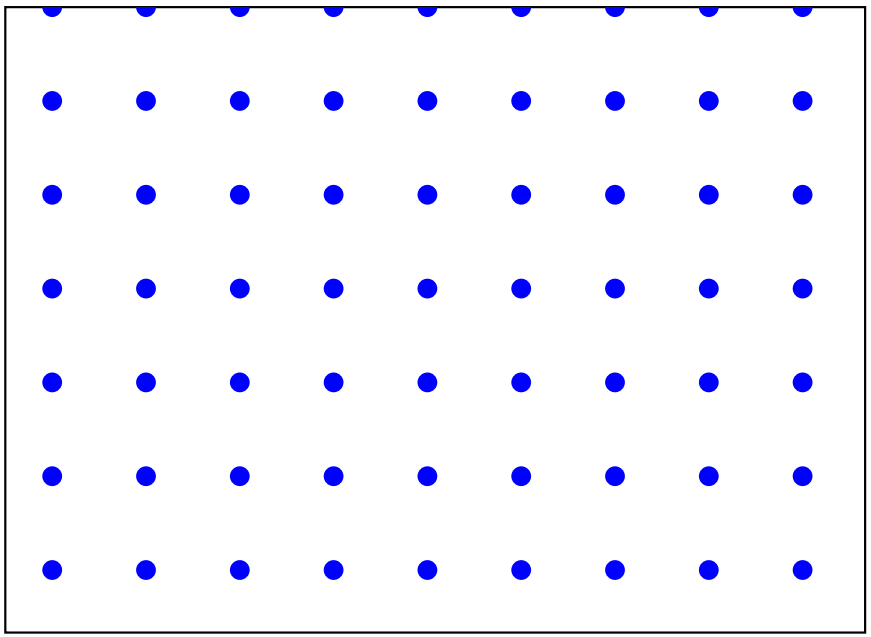
The determinant of a lattice is the volume of a fundamental region

$$\det(\mathbf{B}\mathbb{Z}^n) = \mathrm{vol}_n\left(\mathbf{B}[0,1)^n\right) = \frac{1}{\mathrm{density}(\Lambda)}$$

The Densest Sublattice Problem (DSP)

Definition (Densest Sublattice Problem (k-DSP))

Given a lattice Λ, find a k-dimensional sublattice $\Lambda' \subseteq \Lambda$ that minimizes $\det(\Lambda')$.



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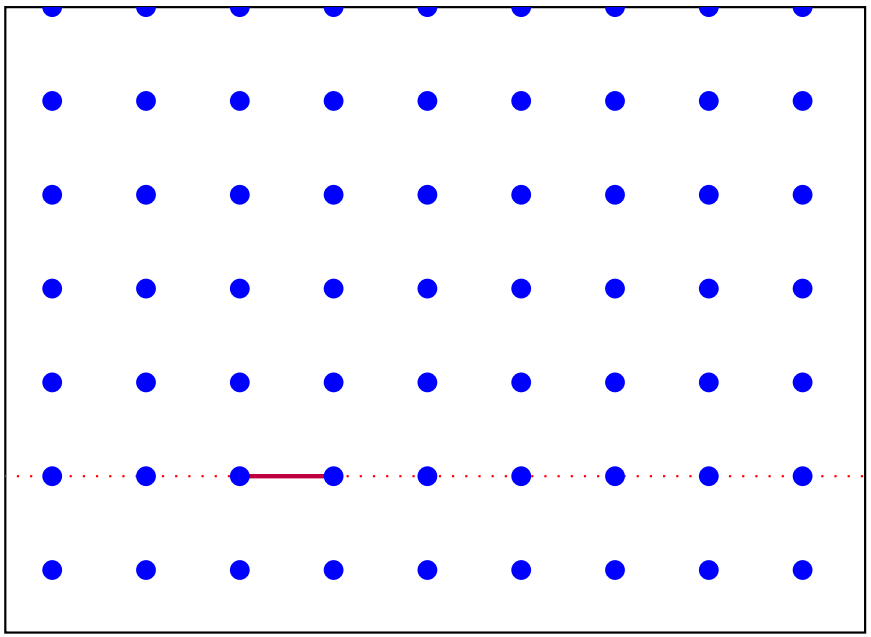

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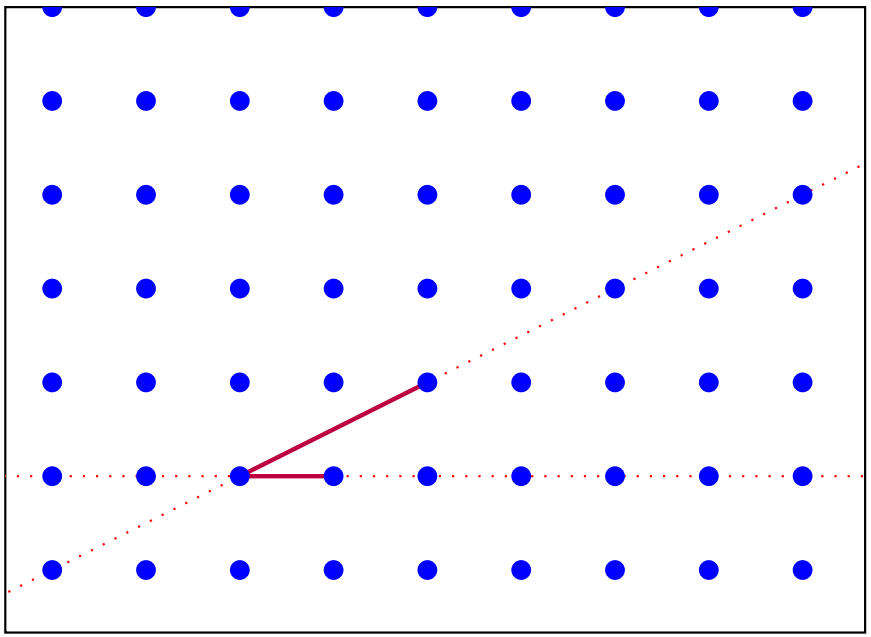

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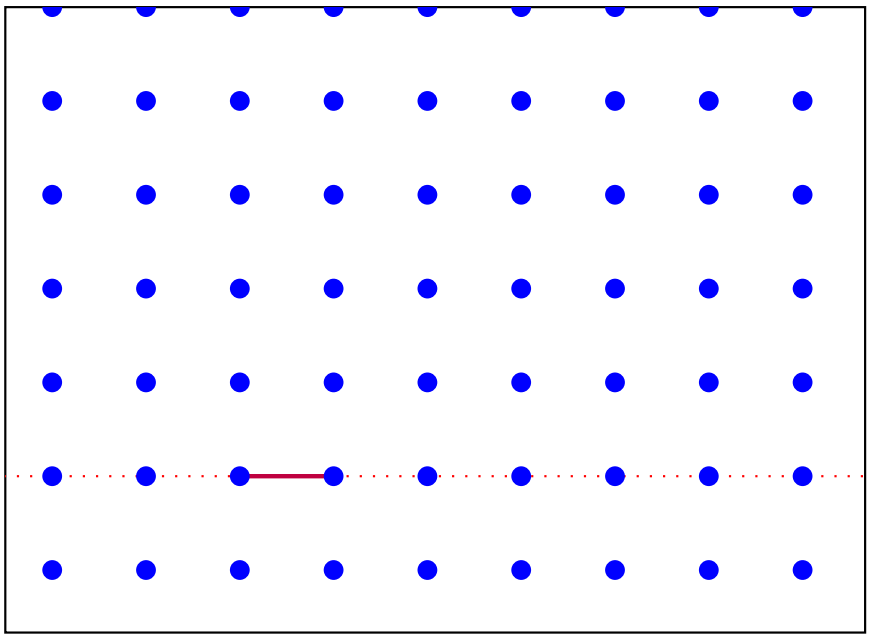

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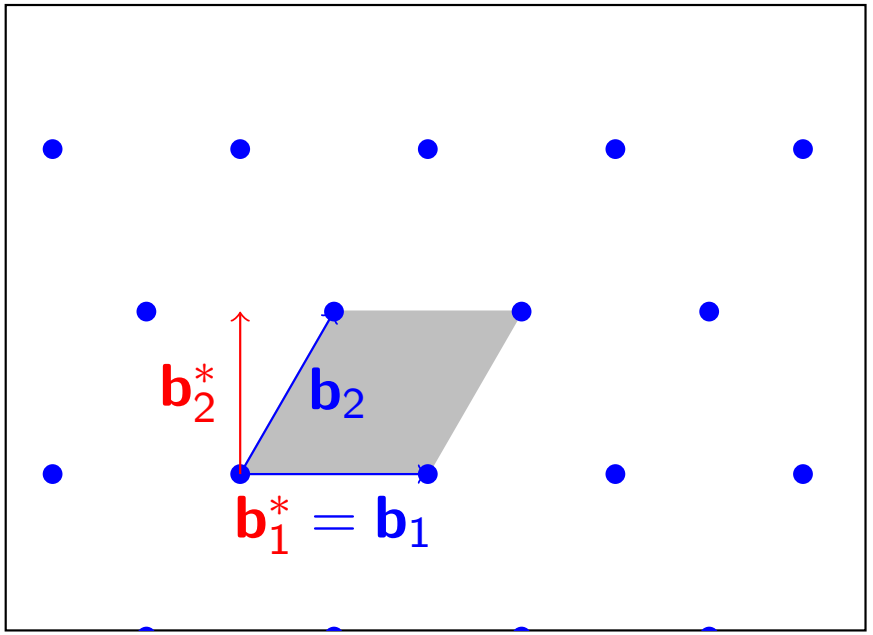
- 1-DSP = SVP
  (Shortest Vector Problem)

Lattice rounding

- Gram-Schmidt orthogonalization $\mathbf{B}^*[0,1)^n$



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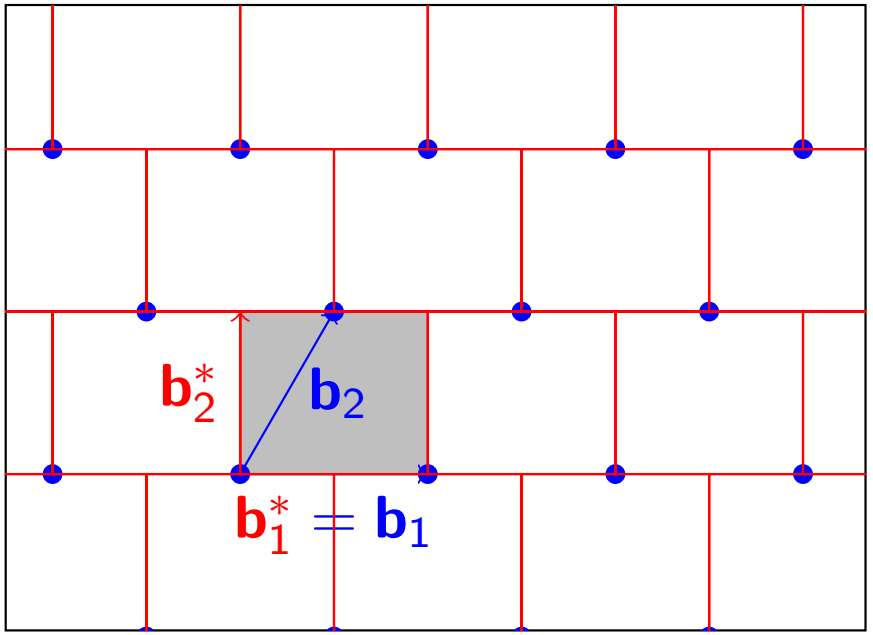

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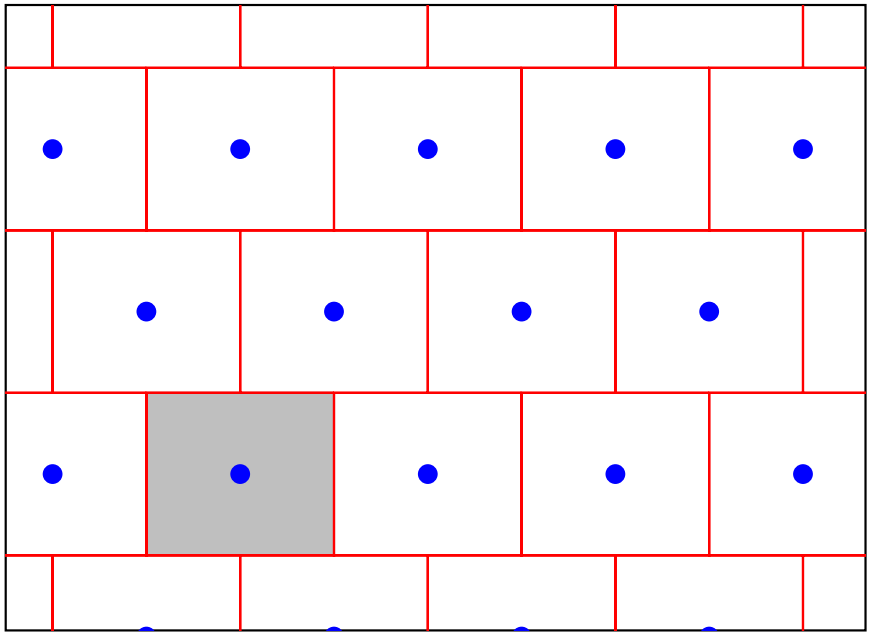

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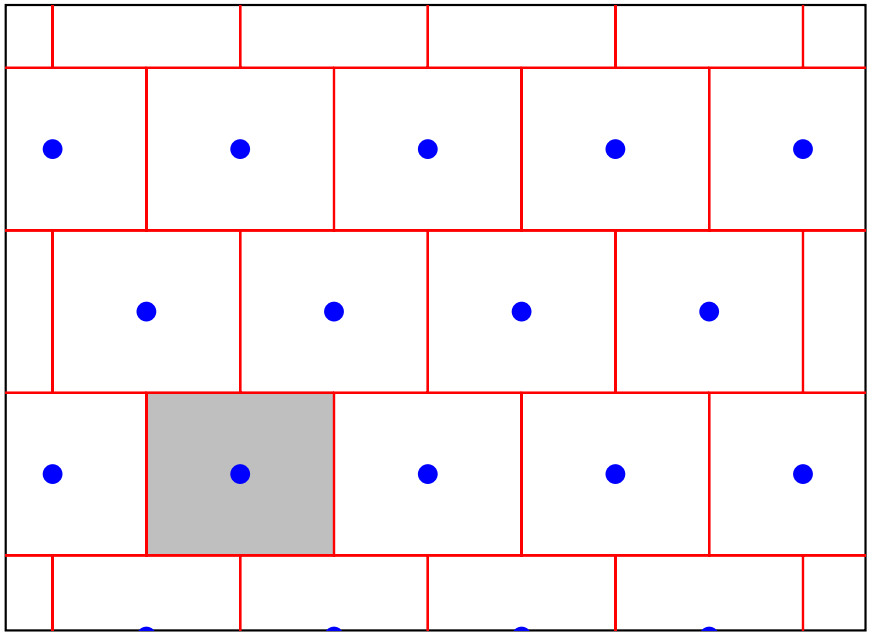

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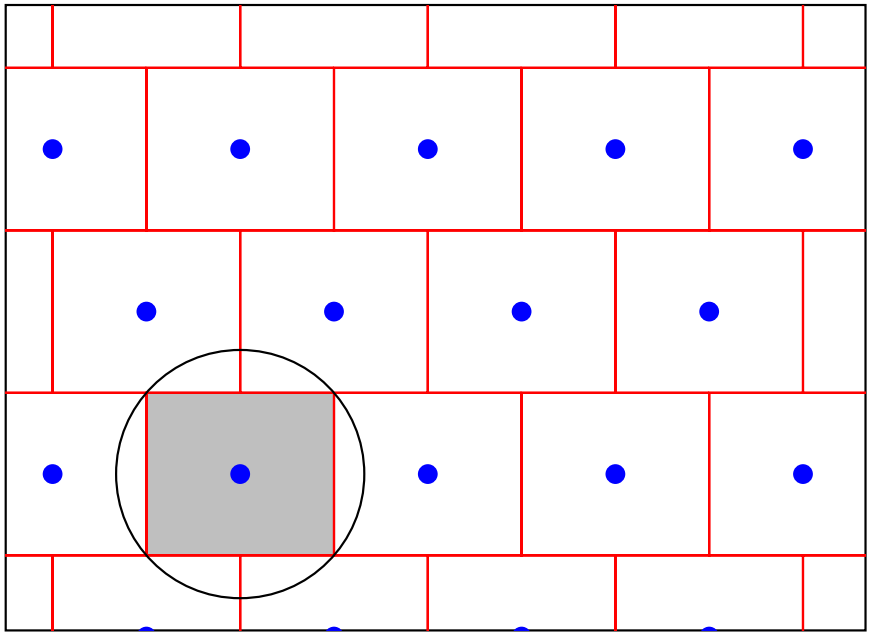

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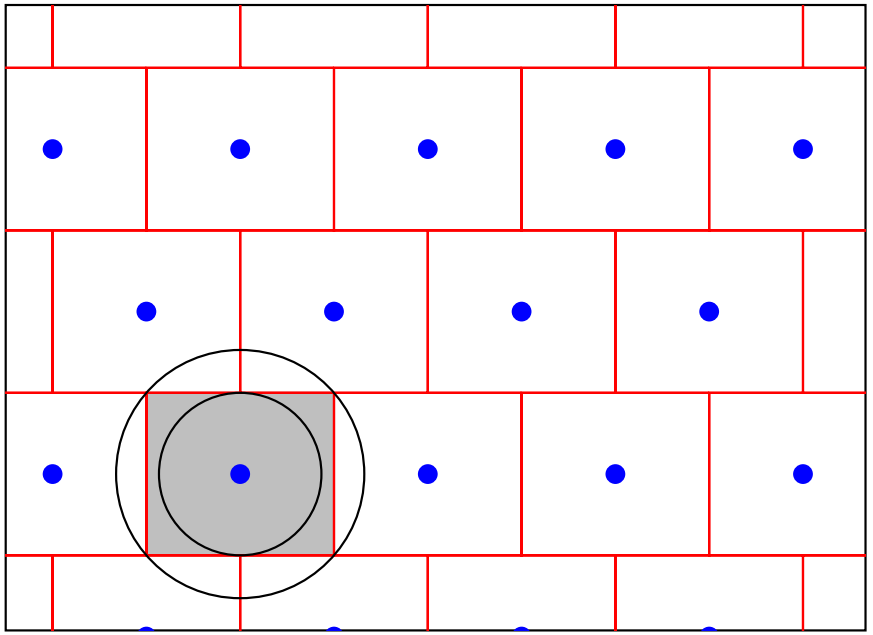

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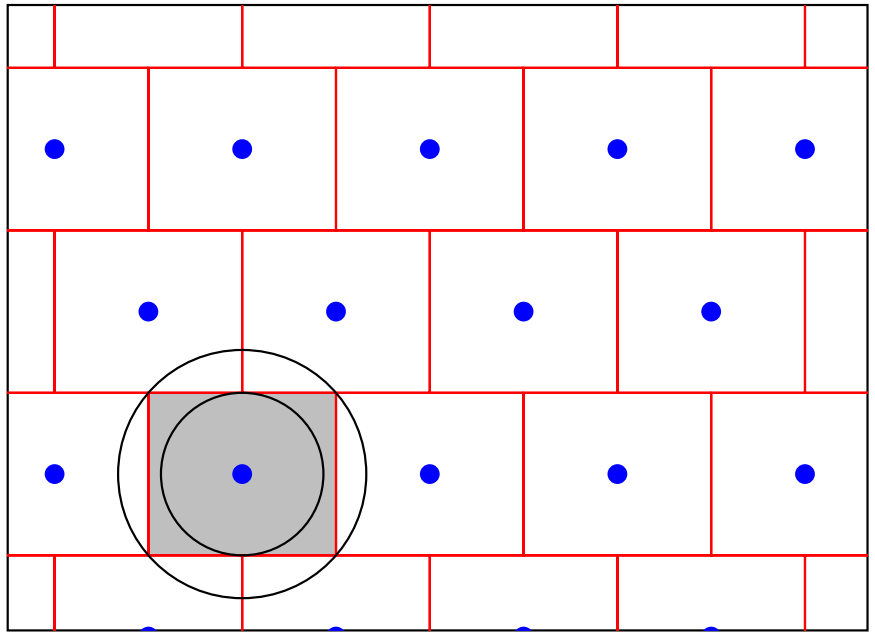

Lemma (Nearest Plane Algorithm [Babai 1986])

Rounding w.r.t $\mathbf{B}^*$ *approximates CVP within* $\sqrt{n} \cdot \frac{\max_i \|\mathbf{b}_i^*\|}{\min_i \|\mathbf{b}_i^*\|}$

Basis reduction

Definition (Basis reduction problem)

Given a lattice, find a basis such that $\|\mathbf{b}_i^*\| \approx \det(\Lambda)^{1/n}$, or, more generally, the $\|\mathbf{b}_i^*\|$ do not decrease too quickly.

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- Hemite constant:

$$\gamma_n = \sup_{\Lambda} \inf_{\mathbf{B}} \left(\frac{\|\mathbf{b}_1\|}{\det(\Lambda)^{1/n}} \right)^2 = \Theta(n)$$

LLL basis reduction algorithm

Theorem (Lenstra, Lenstra, Lovasz (LLL) 1982)

Every lattice has an efficiently computable basis such that $\|\mathbf{b}_{i+1}^*\| \geq \tilde{\gamma}_2 \cdot \|\mathbf{b}_i^*\|$ *for all* i*, and* $\frac{\max_i \|\mathbf{b}_i^*\|}{\min_i \|\mathbf{b}_i^*\|} = 2^{O(n)}$

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- Polynomial time termination for any $\tilde{\gamma}_2 > \gamma_2$

Generalized basis reduction algorithm

- Partition basis vectors into blocks $\mathbf{B} = [\mathbf{B}_1, \ldots, \mathbf{B}_m]$

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- [Gama,Howgrave-Graham,Koy,Nguyen 2006]: $\dim(B_i) = k$

DSP and LLL [GHKN'06]

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- A lattice can have as many as $2^{O(n^3)}$ reduced bases!

Better solutions for DSP?

- Recursive approach to find solution M to k-DSP
  - Minimize $\det(M) = \|\mathbf{b}_1^*\| \cdots \|\mathbf{b}_k^*\|$ where $M = [\mathbf{b}_1, \ldots, \mathbf{b}_k]$
  - First find some $\mathbf{b}_1 \in M$, somehow.
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  - Does not work! E.g., k-DSP for $k = 2, n = 3$:

SVP	$\mathbf{b}_1$	$=$	(a, a, a)	$\sqrt{1/2} < a < \sqrt{2/3}$
2-DSP	$\mathbf{b}_2$	$=$	$(1, -1, 0)$	
	$\mathbf{b}_3$	$=$	$(0, -1, 1)$	

Main Lemma and Algorithm

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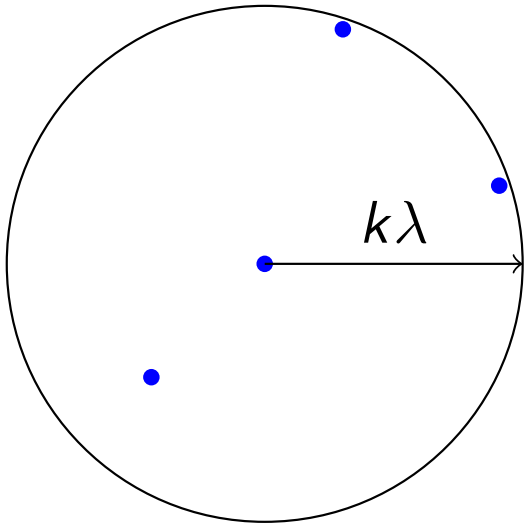
Running time is $T \approx 2^{O(n)} + \binom{L}{k}$

Bounding the list size

- Put a sphere of radius $\lambda/2$ around every point in L
- Spheres are disjoint
- All spheres belong to larger sphere of radius $(k + \frac{1}{2})\lambda$
- Volume bound:

$$|L| \cdot \left(\frac{1}{2}\right)^n \leq \left(k + \frac{1}{2}\right)^n$$



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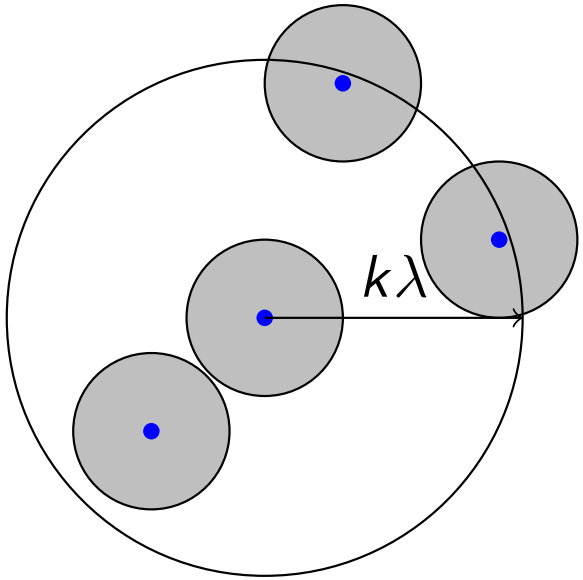

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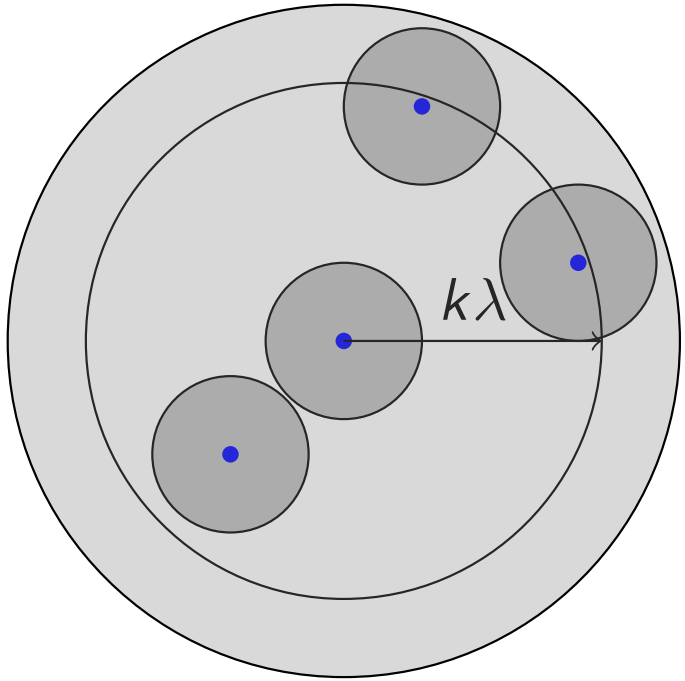

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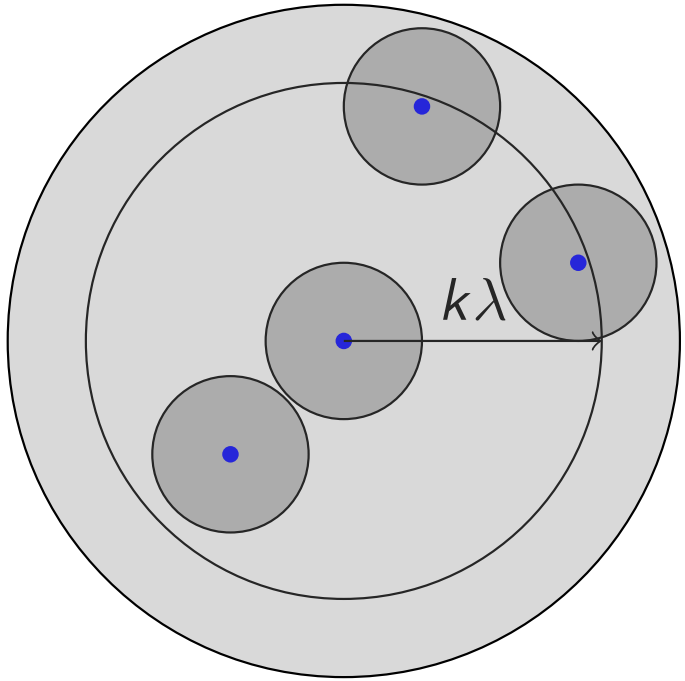

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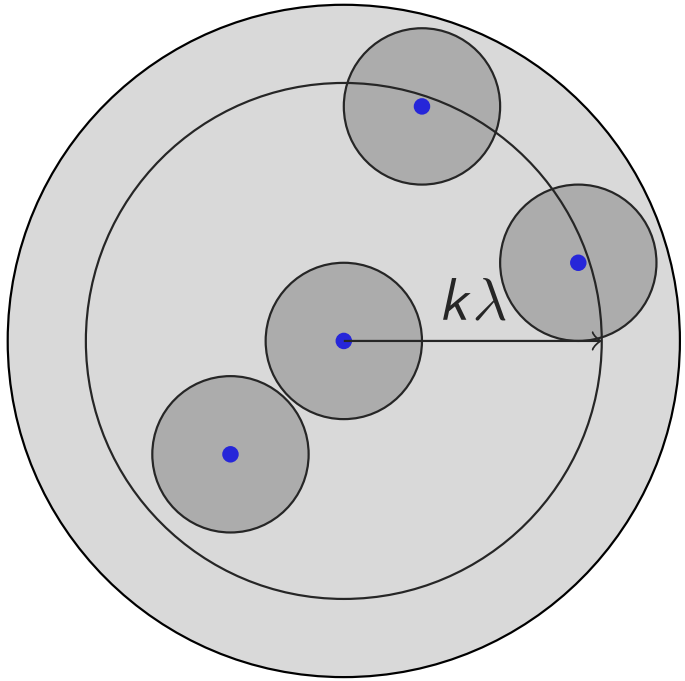

Theorem (Main)

k-DSP can be solved in time

$$T \approx |L|^k = (2k+1)^{kn} = k^{O(kn)}$$

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Thanks! Questions?

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- Even the definition of DSP is not completely obvious

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Open Problem

Develop effective pruning strategies/heuristics to speed up DSP computation, and evaluate performance in practice on random lattices.

Single exponential time?

- Our algorithm solves k-DSP in $TIME(2^{O(n)})$ when $k = O(1)$

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- Alternatively, offer evidence that k-DSP cannot be solved in $2^{O(n)}$, e.g., reduce higher dimensional CVP problem to DSP

Rankin constant (Generalizing Hermite $\gamma_n = \gamma_{n,1}$)

$$\gamma_{n,k} = \sup_{\Lambda} \inf_{\mathbf{B}} \left(\frac{\det(\mathbf{b}_1, \ldots, \mathbf{b}_k)^{1/k}}{\det(\Lambda)^{1/n}} \right)^{2k} \qquad \text{(Rankin 1955)}$$

1	1								
2	$\frac{2}{\sqrt{3}}$	1							
3	$\sqrt[3]{2}$	$\sqrt[3]{2}$	1						
4	$\sqrt{2}$	$3/2$	$\sqrt{2}$	1					
5	$2^{3/5}$			$2^{3/5}$	1				
6	$\sqrt[6]{64/3}$	$3^{2/3}$		$3^{2/3}$	$\sqrt[6]{64/3}$	1			
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- Using "dynamic" h, k may produce even better bases