A PROOF OF ANDREWS' CONJECTURE ON PARTITIONS WITH NO SHORT SEQUENCES

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Abstract. Our main result establishes Andrews' conjecture for the asymptotic of the generating function for the number of integer partitions of n without k consecutive parts. The methods we develop are applicable in obtaining asymptotics for stochastic processes that avoid patterns, as a result they yield asymptotics for the number of partitions that avoid patterns.

Holroyd, Liggett, and Romik, in connection with certain bootstrap percolation models, introduced the study of partitions without k consecutive parts. Andrews showed that when $k = 2$ the generating function for these partitions is a mixed-mock modular form, and, thus has modularity properties which can be utilized in the study of this generating function. For $k > 2$ the asymptotic properties of the generating functions have proved more difficult to obtain. Using q-series identities and the $k = 2$ case as evidence, Andrews stated a conjecture for the asymptotic behavior. We improve upon previous approaches to this problem by identifying and overcoming two sources of error.

1. Introduction and Statement of Results

Studying a generalization of bootstrap percolation (see [1, 12, 14] for examples), Holroyd, Liggett, and Romik [15] introduced the following probability models: Let $0 < s < 1$ and $C_1, C_2, \cdots$ be independent events with probabilities

$$\mathbf{P}_s(C_n) := 1 - e^{-ns}$$

under a probability measure $\mathbf{P}_s$. Let A_k be the event

$$A_k = \bigcap_{i=1}^{\infty} (C_i \cup C_{i+1} \cup \cdots \cup C_{i+k-1})$$

that there is no sequence of k consecutive C_i values that do not occur. The relevant question in [15] is to understand the behavior as $s \downarrow 0$. Theorem 2 of Holroyd, Liggett, and Romik [15] gives

$$\log\left(\mathbf{P}_s(A_k)\right) \sim -\frac{L_k}{s}$$

where

$$L_k := \frac{\pi^2}{3k(k+1)}. \tag{1.1}$$

Obtaining an estimate for $\mathbf{P}_s(A_k)$ with polynomial relative error has proven to be a challenging problem. Bringmann and Mahlburg [8] refined the result of Holroyd, Liggett, and

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Romik by proving non-logarithmic lower and upper bounds that differed by a polynomial factor of $s^{-\frac{1}{k}}$. Precisely, they give (Theorem 1.2 [8])

$$\exp\left(-\frac{\lambda_k}{s}\right) \ll_k \mathbf{P}_s(A_k) \ll_k s^{-\frac{2k-1}{2k}} \exp\left(-\frac{\lambda_k}{s}\right).$$

With Mellit, Bringmann and Malburg [9] developed a general method for establishing similar bounds for natural families of pattern-avoiding sequences. Underlying all of the above results are estimates for the eigenvalues of an associated (Markov-type) stochastic process.

Andrews [3] established a surprising connection between $\mathbf{P}_s(A_2)$ and one of Ramanujan's mock theta functions. Precisely, he showed

$$\mathbf{P}_s(A_2) = \prod_{n=1}^{\infty} \frac{1+q^{3n}}{1+q^n} \cdot \chi(q) \tag{1.2}$$

where $q := e^{-s}$ and $\chi(q) = \sum_{n=0}^{\infty} q^{n^2} \prod_{m=1}^{n} \frac{1+q^m}{1+q^{3m}}$ is a mock theta function. Zwegers' Ph.D. thesis [22] yields the modular properties of Ramanujan's mock theta functions (see [20] or [19] for details). Consequentially, Andrews [3], using (1.2) and additional identities for Ramanujan's mock theta functions proved that

$$\mathbf{P}_s(A_2) \sim \sqrt{\frac{\pi}{2}} s^{-\frac{1}{2}} \exp\left(-\frac{\pi^2}{18s}\right) \quad \text{as} \quad s \downarrow 0.$$

Using additional q-series identities when $k > 2$, he made the following conjecture.

Conjecture 1.1 (Andrews [3]). *For each $k \geq 2$, there exists a positive constant D_k such that*

$$\mathbf{P}_s(A_k) \sim D_k s^{-\frac{1}{2}} \exp\left(-\frac{\lambda_k}{s}\right) \quad \text{as } s \downarrow 0.$$

We prove the following precise version of Andrews's Conjecture.

Theorem 1.2. *Andrews's conjecture is true with $D_k = \frac{\sqrt{2\pi}}{k}$. More specifically, we have*

$$\mathbf{P}_s(A_k) = \frac{\sqrt{2\pi}}{k} s^{-\frac{1}{2}} \exp\left(-\frac{\pi^2}{3k(k+1)s} + O_k\left(s^{\frac{1}{2k+3}}\right)\right).$$

Remark 1.3. *We expect that our techniques can be improved to give a full asymptotic expansion for $\mathbf{P}_s(A_k)$ with relative error $O(s^N)$, for any N.*

1.1. **More on partitions and additional applications.** There is an unexpected and beautiful connection between these models and partitions, Ramanujan's mock theta functions, and the Rogers-Ramanujan identities.

A partition μ of n has a k-sequence if there are k parts of consecutive sizes. Let $p_k(n)$ denote the number of partitions of n with no k-sequences and $G_k(q) := \sum_{n=0}^{\infty} p_k(n) q^n$ the generating function. Set $p_k(0) = 1$. In Section 4 of [15] it is shown that

$$\mathbf{P}_s(A_k) = \frac{G_k(q)}{P(q)} \tag{1.3}$$

when $q := e^{-s}$ and $P(q) = \sum_{n=0}^{\infty} p(n)q^n = \prod_{n=1}^{\infty} \frac{1}{1-q^n}$, where $p(n)$ is the number of partitions of n.

Asymptotics for $P(q)$ are well known, namely

$$P(q) = \frac{1}{\sqrt{2\pi}} s^{\frac{1}{2}} \exp\left(\frac{\pi^2}{6s} - \frac{s}{24} + O(s^N)\right)$$

for any N. Thus, by (1.3), determination of the asymptotics of $G_k(q)$ is equivalent to the determination of the asymptotic of $\mathbf{P}_s(A_k)$. We prove the following theorem, which is equivalent to Theorem 1.2.

Theorem 1.4. *For each $k \geq 2$ we have*

$$G_k(e^{-s}) = \frac{1}{k} \exp\left(\frac{\pi^2}{6s}\left(1 - \frac{2}{k(k+1)}\right) + O_k\left(s^{\frac{1}{2k+3}}\right)\right)$$

as $s \downarrow 0$.

Remark 1.5. *A slight modification of the arguments presented establish Theorem 1.4 with a relative error that is $o(1)$ for non-real s satisfying $|\Im(s)| = o(\Re(s))$.*

Numerical calculations lead to the following conjecture for real s.

Conjecture 1.6. *For s real and $s \downarrow 0$*

$$G_k(e^{-s}) = \frac{1}{k} \exp\left(\frac{\pi^2}{6s}\left(1 - \frac{2}{k(k+1)}\right) + \alpha_k s^{\frac{1}{k}} + O\left(s^{\frac{2}{k}}\right)\right)$$

for some constant α_k.

Remark 1.7. *Classical results on the mock theta functions* [3] *easily yield this conjecture for $k = 2$ with $\alpha_2 = \sqrt{\frac{2}{9\pi}}$. Using a numerical technique, Zagier* [21] *has calculated that $\alpha_3 \approx 0.26627104041\ldots$, which appears to equal $\frac{3^{1/3}}{4\Gamma(2/3)}$. It is amusing to note that $\alpha_2 = \sqrt{\frac{2}{9\pi}} = 0.265961520\ldots$ agrees with α_3 to two digits.*

This conjecture implies that for $k > 2$ the generating function $G_k(q)$ is not a usual modular form. Indeed, if $G_k(q)$ is a half integral weight modular form or mixed mock modular form, we would expect an asymptotic expansion that contains only powers of $s^{\frac{1}{2}}$. In fact, the asymptotic expansion in powers of $s^{1/k}$ is very exotic.

It is well known that the asymptotic behavior of generating functions leads to asymptotics for the coefficients. We obtain the following theorem for the asymptotic of $p_k(n)$.

Theorem 1.8. *As $n \to \infty$ we have*

$$p_k(n) \sim \frac{1}{2k}\left(\frac{1}{6}\left(1 - \frac{2}{k(k+1)}\right)\right)^{\frac{1}{4}} \frac{1}{n^{\frac{3}{4}}} \exp\left(\pi\sqrt{\frac{2}{3}\left(1 - \frac{2}{k(k+1)}\right)n}\right).$$

Remark 1.9. *Bringmann and Mahlburg* [7] *use the connection with Ramanujan's mock theta function and an extension of the circle method to prove a nearly exact formula for $p_2(n)$.*

While the study of partitions without k-sequences for $k > 2$ is relatively new, there are several classical results on partitions without 2-sequences. The Rogers-Ramanujan identities state that

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(1-q)\cdots(1-q^n)} = \prod_{n=1}^{\infty} \frac{1}{(1-q^{5n-4})(1-q^{5n-1})}, \tag{1.4}$$

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(1-q)\cdots(1-q^n)} = \prod_{n=1}^{\infty} \frac{1}{(1-q^{5n-3})(1-q^{5n-2})}. \tag{1.5}$$

MacMahon [18] found a combinatorial interpretation of Rogers-Ramanujan identities as a way of counting partitions without 2-sequences with some particular constraints. In particular, he shows that they imply:

(1.4)* The partitions of n into distinct parts with no parts of consecutive size are equinumerous with the number of partitions of n into parts of the form $5n-4$ and $5n-1$.

(1.5)* The partitions of n into distinct parts with no parts of consecutive size and no parts of size 1 are equinumerous with the number of partitions of n into parts of the form $5n-3$ and $5n-2$.

This combinatorial interpretation is important in the hard hexagon model studied by Baxter [6]. Another use of these identities is that the product expansions reveal that these series in question are essentially modular forms and thus their analytic nature is well understood. For example, with $q = e^{-s}$, for any A, the series in (1.4) satisfies

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(1-q)\cdots(1-q^n)} = \sqrt{\frac{2}{5-\sqrt{5}}} \exp\left(\frac{\pi^2}{15s} - \frac{s}{60}\right) + O(s^N) \quad \text{as} \quad s \downarrow 0.$$

It is surprising that the generating functions for partitions without consecutive parts often have product expansions resembling those of the Rogers-Ramanujan identities. Let $p_{k,r,>B}(n)$ be the number of partitions of n with no k parts of consecutive sizes, no part occurring more than r times, and no parts of size $\leq B$. Then (1.4) and (1.5) are identities for the generating functions $\sum_{n=0}^{\infty} p_{2,1,>0}(n)q^n$ and $\sum_{n=0}^{\infty} p_{2,1,>1}(n)q^n$. We have the following partition identities:

$$\sum_{n=0}^{\infty} p_{2,2,>1}(n)q^n = \prod_{n=1}^{\infty} \frac{1}{(1-q^{6n-2})(1-q^{6n-3})(1-q^{6n-4})}$$

$$\sum_{n=0}^{\infty} p_{2,2,>0}(n)q^n = \prod_{n=1}^{\infty} \frac{(1-q^{6n-3})^2(1-q^{6n})}{(1-q^n)}$$

$$\sum_{n=0}^{\infty} p_{2,\infty,>1}(n)q^n = \prod_{n=1}^{\infty} \frac{1}{(1-q^{6n})(1-q^{6n-2})(1-q^{6n-3})(1-q^{6n-4})}$$

The first identity is due to Andrews [2], the second identity is due to MacMahon [18] and the final identity is due to Andrews and Lewis [5]. The above identities along with modular form techniques allow for very precise asymptotics for the generating functions of partitions that avoid 2-sequences in addition to satisfying some additional constraints. Unfortunately, these techniques have failed for partitions with no k-sequences for $k > 2$. This may be because, as remarked above, Conjecture 1.6 would imply that the corresponding generating functions are not modular.

Due to (1.3) studying $G_k(q)$ is equivalent to studying $\mathbf{P}_s(A_k)$ when $q = e^{-s}$. This equivalence provides two equivalent languages in which to discuss our results, that of partitions and that of probability. Throughout most of this paper we will use the former language to discuss our techniques. There are three reasons for this somewhat arbitrary choice. Firstly, the discussion of the behavior of the small parts of the partition (or equivalently the C_i for small i) fits slightly more naturally into this language. Secondly, our own backgrounds are in combinatorics. Finally, numerical calculations suggest the asymptotics of the function $G_k(e^{-s})$ is given purely in powers of $s^{\frac{1}{k}}$, whereas the asymptotics of $\mathbf{P}_s(A_k)$ will have an extra $s^{\frac{1}{2}}$ multiplying all terms.

1.2. **The Approach.** In this section, we sketch the proof of Andrews's conjecture. The fundamental idea is to compute $G_k(q)$ as the limiting value of a recurrence relation. In particular, one can imagine building a partition by adding parts of one size at a time: first determining the number of parts of size one, then the number of parts of size two and so on. In order to ensure that the partition constructed has no k-sequences, one would need to keep track of how many of the recent part sizes have been used and ensure that no k sizes in a row are employed. In order to keep track of the necessary generating functions, we define

$$\widetilde{v}_i^k(N,q) := \sum_{\substack{\mu \text{ a partition with parts } \leq N \\ \mu \text{ has no } k \text{ parts with consecutive sizes} \\ \mu \text{ has parts of size } N, N-1, \cdots, N-i+1 \\ \mu \text{ has no part of size } N-i}} q^{|\mu|}. \tag{1.6}$$

In particular, we note that for $N = 0$ we have that

$$\widetilde{v}_i^k(0,q) = \begin{cases} 1 & \text{if } i = 0 \\ 0 & \text{otherwise} \end{cases}.$$

We have the following recursion

$$\begin{pmatrix} \widetilde{v}_0^k(N,q) \\ \widetilde{v}_1^k(N,q) \\ \vdots \\ \widetilde{v}_{k-1}^k(N,q) \end{pmatrix} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ z(q^N) & 0 & \cdots & 0 \\ 0 & z(q^N) & \cdots & 0 \\ 0 & & \vdots & 0 \\ 0 & \cdots & z(q^N) & 0 \end{pmatrix} \begin{pmatrix} \widetilde{v}_0^k(N-1,q) \\ \widetilde{v}_1^k(N-1,q) \\ \vdots \\ \widetilde{v}_{k-1}^k(N-1,q) \end{pmatrix}$$

where $z(x) := \frac{x}{1-x}$. For convenience set

$$m(x) := \begin{pmatrix} 1 & 1 & \cdots & 1 \\ z(x) & 0 & \cdots & 0 \\ 0 & z(x) & \cdots & 0 \\ 0 & & \vdots & 0 \\ 0 & \cdots & z(x) & 0 \end{pmatrix}, \tag{1.7}$$

and

$$v^k(N,q) := \begin{pmatrix} \widetilde{v}_0^k(N,q) \\ \widetilde{v}_1^k(N,q) \\ \vdots \\ \widetilde{v}_{k-1}^k(N,q) \end{pmatrix}. \tag{1.8}$$

Thus, we have the recursion

$$v^k(N,q) = m(q^N)v^k(N-1,q).$$

Furthermore, it is not hard to see that

$$G_k(q) = \lim_{N\to\infty} \widetilde{v}_0^k(N,q).$$

Thus, we have a linear, homogeneous recurrence relation with non-constant coefficients whose limiting value yields $G_k(q)$. The main idea for evaluating this quantity is as follows. If the matrices, $m(q^N)$, were constant (or even merely simultaneously diagonalizeable), the product would be easy to evaluate and $G_k(q)$ would be approximately equal to the product of the largest eigenvalues. This is not the case, but fortunately, the matrices $m(q^n)$ vary slowly with n. The difficulty in approximating $G_k(q)$ comes in figuring out how to take advantage of this.

This basic approach is not new to this problem. Holroyd, Liggett, and Romik [15] implicitly employ a similar recurrence relation to obtain $\mathbf{P}_s(A_k)$. Since the $m(q^n)$ are slowly varying in n, they approximate products of roughly $s^{-1/2}$ of these matrices by making the approximation that all of the matrices in the block are the same. Within each of these blocks, standard eigenvalue techniques are used to evaluate the product. This technique allows for asymptotic approximation of $\log(\mathbf{P}_s(A_k))$, yielding a term coming from the product of the largest eigenvalues of $m(q^n)$, but has two major sources of error. The first of these errors comes from the approximation that each of the $m(q^n)$ within a block are constant. This is especially problematic for the early blocks, for which $m(q^n)$ is rapidly varying with n. The second source of error comes from having poor control over the transitions between blocks. Our technique avoids these difficulties, but requires new ideas to approximate this product of non-commuting matrices.

Our main idea is to write the vectors $v^k(N,q)$ in terms of the slowly varying eigenbasis of the matrices $m(q^n)$. In particular, we may diagonalize each matrix as

$$m(q^n) = A(q^n)D(q^n)A(q^n)^{-1}$$

where $D(q^n)$ is the diagonal matrix with $\lambda_1(q^n)$, the primary eigenvalue of $m(q^n)$, in the upper-left hand corner of the matrix. Thus, in the appropriate basis, multiplying by m, corresponds to multiplication by the diagonal matrix D. Unfortunately, in order to rewrite $v^k(n,q)$ in terms of the appropriate eigenbasis for $m(q^{n+1})$, we must also multiply by the transition matrix

$$T(n,q) = A(q^{n+1})^{-1}A(q^n).$$

Since, the coefficients of $A(q^n)$ are slowly varying, $T(n,q) \approx I_k$, where I_k is the $k\times k$ identity matrix. In particular, with $q = e^{-s}$ we establish $T(n,q) = I_k + O(n^{-1}+s)$. As n becomes large, the primary eigenvalue becomes much larger than the others and so multiplying by

$D(q^n)$ decreases the sizes of the other coordinates relative to the first coordinate. Consequentially, the vector of interest is well approximated by the first coordinate. Ignoring the off-diagonal entries of the $T(n,q)$, we find that $G(q)$ is roughly

$$\prod_n \lambda_1(q^n) \prod_n T(n,q)^{1,1},$$

where $T(n,q)^{1,1}$ is the upper-left hand entry of $T(n,q)$. The product $\prod_n \lambda_1(q^n)$ is handled through an analysis of the characteristic polynomial of $m(q^n)$ by Holroyd, Liggett, and Romik [15]. However, we require a refinement of their calculations to obtain sufficient errors (see Theorem 5.1). The product of the transition matrix entries is similar to, but more delicate than, the analysis used to compute $\prod \lambda_1(q^n)$ (see Theorem 4.4).

A second new idea is needed to deal with the contribution of matrices with N small. For small n the main eigenvector of $m(q^n)$ is not a good approximation for the contribution to the generating function. In fact, the non-primary eigenvalues contribute to the asymptotic approximation. To overcome this difficulty we use a direct combinatorial analysis to approximate $v^k(n,q)$ for small n. This analysis appears in Section 3.

We note some similarities between this technique and the adiabatic approximation in quantum mechanics (see, for example, Chapter 10 of [13]). In each case, we are sequentially applying a sequence of slowly-varying matrices to a given initial vector (though in the adiabatic process, this is done continuously rather than discretely). In each case, we write our vectors in terms of the (slowly changing) eigenbasis. The final outcome is approximated by taking the product (or integral) of the eigenvalues, with a correction term due to the change of basis (known as Berry's phase in the case of quantum mechanics). The justifications for this approximation are different in the two cases, for while the adiabatic approximation holds due to cancelation of cross terms due to rapid oscillation, in our case the approximation holds because the contribution from the non-primary eigenvectors may be safely neglected.

It should be noted that our underlying ideas have far more general applicability than simply to the problem at hand. In particular, we expect that they can be used to calculate the asymptotics of the number of partitions that locally "avoid patterns" of various types (for example, not having any k parts of consecutive sizes). For example, we believe that our techniques should be able to prove asymptotics for $p_{k,r,>B}(n)$ for all k, r, B. As an additional example, Knopfmacher and Munagi [17] consider the problem of counting the number of partitions $\lambda = (\lambda_1, \cdots, \lambda_\ell)$ of n such that there is no j with $\lambda_j - \lambda_{j+1} = p$ for any fixed $p > 0$. The methods in this paper should also be sufficient to approximate the number of partitions of these types, although the constants showing up in the asymptotic formulae may well not have closed forms.

Bringmann, Mahlburg, and Mellit introduced a family of directed, multi-state bootstrap percolation models [9]. Their study led to the following: Let $\{E_j\}_{j=1}^n$ be a sequence of random variables taking values in $\{A, B, C, D\}$ such that

$$\mathbf{P}_s(E_j = A) = (1-e^{-js})^2, \;\; \mathbf{P}_s(E_j = B) = \mathbf{P}_s(E_j = C) = e^{-js}(1-e^{-js}), \;\; \mathbf{P}_s(E_j = D) = e^{-2js}.$$

They are interested in the behavior of

$$\mathbf{P}_s\left(\{E_j\}_{j=1}^n \text{ has no } D, CB, \text{ or } C^k\right)$$

where C^k denotes a sequence of k consecutive Cs, as $s \downarrow 0$. Surprisingly, similar to Andrews identity (1.2), they found a connection with a mock theta function in the case $k = 2$. Again, our methods should yield an asymptotic for this probability as $s \downarrow 0$ with a relative error which is polynomial in $s^{1/k}$.

1.3. **Structure of the paper.** As discussed above, our argument splits into two main pieces. On the one hand, we need a direct way of computing $v^k(n, q)$ for small values of n. Then, once we have gotten to the point where $v^k(n, q)$ is well approximated by the primary eigenvector of $m(q^n)$, we can use the recurrence relation described above.

In Section 2, we perform some preliminary calculations involving the $m(q^n)$ and their eigenvalues that will be used throughout. Section 3 gives a direct computation for the generating functions $v^k(N, q)$ for N of size $s^{-\frac{1}{k+1}-\epsilon}$. In Section 4, we analyze the recurrence relation in order to compute $G_k(q)$. Section 5 contains an estimate for the product over the largest eigenvalues. Section 6 gives the proof of Theorem 1.4 and thus Theorem 1.2. Section 7 gives the proof of Theorem 1.8.

2. Calculations on the Diagonalization of $m(q^n)$

In this section, we collect some results on the eigenvalues and diagonalization of the matrices $m(q^n)$. In this section, k is fixed and s is assumed to be small. Errors are often written in big-O notation. In almost all cases the constants depend on k. We often suppress this dependence inside of the proofs.

Observe that the characteristic polynomial of $\frac{1}{z(q^n)} m(q^n)$ is

$$\lambda^k - z(q^n)^{-1} \left(\lambda^{k-1} + \cdots + \lambda + 1\right).$$

We begin by proving some basic results about the sizes of the eigenvalues of this polynomial when $z(q^n)$ is either very large or very small.

Lemma 2.1. *For $z \in \mathbb{R}$, let $\lambda_i(z)$ be the roots of $\lambda^k - z^{-1}\left(\lambda^{k-1} + \cdots + \lambda + 1\right) = 0$. Then for z large,*

$$\lambda_i(z) = \omega_i z^{-1/k} \left(1 + \frac{\omega_i}{k} z^{-1/k} + O_k\left(z^{-\frac{2}{k}}\right)\right)$$

where the ω_i are the distinct k^{th} roots of unity. Furthermore, for z small one root satisfies

$$\lambda_i = z^{-1}(1 + O_k(z)),$$

and all other roots satisfy

$$\lambda_i = \omega_i(1 + O_k(z)),$$

where the ω_i here are distinct k^{th} roots of unity other than 1.

Proof. For the first statement, note that we only need to show this for $z \gg 1$. We claim that $p(\lambda) = \lambda^k - z^{-1}(\lambda^{k-1} + \cdots + 1)$ has a root within $O(z^{-2/k})$ of $z^{-1/k}\omega$ for every k^{th} root of unity ω. This follows easily noting that $p(z^{-1/k}\omega) = O(z^{-(k+1)/k})$, $|p'(z^{-1/k}\omega)| = \Theta(z^{-(k-1)/k})$ and that $|p^{(\ell)}(z^{-1/k}\omega)| = O(z^{-(k-\ell)/k})$. This gives $\lambda_i = \omega_i z^{-\frac{1}{k}}\left(1 + O(z^{-\frac{1}{k}})\right)$. The stronger claim follows from

$$\lambda_i^k = z^{-1}\left(1 + \lambda_i + O\left(z^{\frac{2}{k}}\right)\right).$$

For the later two claims, we note that it suffices to consider $z \ll 1$. For the second claim we note that $|p(z^{-1})| = O(z^{-k+1})$, $|p'(z^{-1})| = \Theta(z^{-k+1})$ and $|p^{(\ell)}(z^{-1})| = O(z^{-k+\ell})$. For the final claim, note that if ω is a root of $x^{k-1} + \ldots + 1$ that $|p(\omega)| = O(1)$, $|p'(\omega)| = \Theta(z^{-1})$ and $|p^{(\ell)}(\omega)| = O(z^{-1})$. □

Lemma 2.2. *For every positive real z, the polynomial $p(\lambda) = \lambda^k - z^{-1}\left(\lambda^{k-1} + \cdots + \lambda + 1\right)$ has no repeated roots.*

Proof. Note that if λ is a double root of p then it is a double root of $p(x)(x-1) = x^{k+1} - (1+z^{-1})x^k + z^{-1} = 0$ and therefore a root of the derivative of this, namely, $\left((k+1)x - (1+z^{-1})k\right)x^{k-1}$. Since $x = 0$ is clearly not a root of p, we have that the double root must be $x = k(1+z^{-1})/(k+1)$. On the other hand, it is clear from the form of p, that there is a unique, non-repeated positive real root.

□

Definition 2.3. By Lemma 2.2, the roots of $\lambda^k - z(q^n)^{-1}\left(\lambda^{k-1} + \cdots + \lambda + 1\right)$ are distinct for any n and s. Therefore, the eigenvalues can be analytically continued to functions of $n \in \mathbb{R}^+$. By Lemma 2.1, as $s \to 0$, the various eigenvalues are asymptotic to $e^{\frac{2\pi i j}{k}} z^{-\frac{1}{k}}$. We let $\lambda_j(q^n)$ denote the root whose analytic continuation is asymptotic to $e^{2\pi i\frac{(j-1)}{k}} z(q^n)^{-\frac{1}{k}}$. Thus $\lambda_1(q^n)$ is the unique positive real root of this polynomial. We note that $\lambda_j(q^n)z(q^n)$ are the eigenvalues of $m(q^n)$ and we call $\lambda_1(q^n)z(q^n)$ the *primary eigenvalue* of the matrix $m(q^n)$.

Remark 2.4. *This notation differs slightly from that of Section 1.2. It is convenient for us to separate out the factor of $z(q^n)$ from the eigenvalue.*

Since there are no repeated roots of the characteristic polynomial of $m(q^n)$ for each eigenvalue $z\lambda_j = z(q^n)\lambda_j(q^n)$ of $m(q^n)$ we have the eigenvector $V_n^j := \begin{pmatrix} 1 \\ \lambda_j^{-1} \\ \vdots \\ \lambda_j^{-k+1} \end{pmatrix}$. So we have

$$m(q^n) = A(q^n)D(q^n)A(q^n)^{-1} \tag{2.1}$$

with

$$D = D(q^n) = \begin{pmatrix} z\lambda_1 & 0 & \cdots & 0 \\ 0 & z\lambda_2 & \cdots & 0 \\ & & \vdots & \\ 0 & 0 & \cdots & z\lambda_k \end{pmatrix} \tag{2.2}$$

and

$$A = A(q^n) = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1^{-1} & \lambda_2^{-1} & \cdots & \lambda_k^{-1} \\ & \vdots & \vdots & \\ \lambda_1^{-k+1} & \lambda_2^{-k+1} & \cdots & \lambda_k^{-k+1} \end{pmatrix}. \tag{2.3}$$

Next we turn to the transition matrices $A(q^{n+1})^{-1}A(q^n)$.

Lemma 2.5. *Let* $\lambda_i = \lambda_i(q^{n+1})$ *and* $\mu_i = \lambda_i(q^n)$*, then* $A(q^{n+1}) = \left(\lambda_j^{1-i}\right)_{i,j}$ *and* $A(q^n) = \left(\mu_j^{1-i}\right)_{i,j}$ *and*

$$T(n,q) = (T(n,q)^{i,j})_{i,j} := A(q^{n+1})^{-1}A(q^n) = \left(\prod_{m\neq i}\left(\frac{\mu_j - \lambda_m}{\lambda_i - \lambda_m}\cdot\frac{\lambda_i}{\mu_j}\right)\right)_{i,j} \tag{2.4}$$

where $i = 1, 2, \cdots, k$ *indexes the row and* $j = 1, 2, \cdots, k$ *indexes the column of* $T(n,q)$.

Proof. Note that

$$(A(q^{n+1})^{-1}A(q^n))^T = A(q^n)^T(A(q^{n+1})^{-1})^T.$$

Furthermore,

$$A(q^n)^T\begin{pmatrix} a_0\\ a_1\\ \vdots\\ a_{k-1}\end{pmatrix} = \begin{pmatrix} p(\mu_1^{-1})\\ p(\mu_2^{-1})\\ \vdots\\ p(\mu_{k-1}^{-1})\end{pmatrix}$$

where $p(x) = a_0 + a_1x + \ldots + a_{k-1}x^{k-1}$. Similarly,

$$A(q^{n+1})^T\begin{pmatrix} a_0\\ a_1\\ \vdots\\ a_{k-1}\end{pmatrix} = \begin{pmatrix} p(\lambda_1^{-1})\\ p(\lambda_2^{-1})\\ \vdots\\ p(\lambda_{k-1}^{-1})\end{pmatrix}.$$

Therefore, the (i,j) entry of $A(q^{n+1})^{-1}A(q^n)$ is

$$\mathbf{e}_j^TA(q^n)^T(A(q^{n+1})^{-1})^T\mathbf{e}_i$$

where $\mathbf{e}_i$ is the vector with a 1 in the ith position and zeroes in all others. This, in turn, is the value at λ_j^{-1} of unique degree $(k-1)$ polynomial $p(x)$ so that $p(\lambda_\ell^{-1}) = \delta_{\ell,i}$. Therefore,

$$p(x) = \prod_{m\neq i}\frac{x-\lambda_m^{-1}}{\lambda_i^{-1}-\lambda_m^{-1}}.$$

Thus the (i,j) entry is

$$p(\mu_j^{-1}) = \prod_{m\neq i}\frac{\mu_j^{-1}-\lambda_m^{-1}}{\lambda_i^{-1}-\lambda_m^{-1}} = \prod_{m\neq i}\left(\left(\frac{\mu_j-\lambda_m}{\lambda_i-\lambda_m}\right)\left(\frac{\lambda_i}{\mu_j}\right)\right).$$

□

We will require some lemmas when dealing with transition matrices.

Lemma 2.6. *If* $\lambda_1, \cdots, \lambda_k$ *are the roots of* $\lambda^k - z^{-1}(\lambda^{k-1} + \cdots + \lambda + 1) = 0$ *then we have*

$$|\lambda_i - \lambda_j| \gg_k |\lambda_j|.$$

Proof. By Lemma 2.1 for $|z| \gg 1$, the λ_i are proportional to distinct k^{th} roots of unity, and thus the result follows for $z > C$ for some constant C.

By Lemma 2.1 for $|z| \ll 1$, all but λ_1, are near distinct k^{th} roots of unity, and λ_1 is roughly z^{-1}. Thus if $i = 1$ or $j = 1$, then $|\lambda_i - \lambda_j| \gg z^{-1} \gg |\lambda_j|$. Otherwise, $|\lambda_i - \lambda_j| \gg 1 \gg |\lambda_j|$. Thus the result holds for $z < c$ for some constant c.

For $c \leq z \leq C$, we note that $\frac{\lambda_j}{\lambda_i - \lambda_j}$ is a continuous function of z, and thus has some absolute upper bound. Thus the Lemma holds in this range as well. □

Lemma 2.7. *In the notation of Lemma 2.5, for any i and n we have $|\mu_i - \lambda_i| = O_k(|\lambda_i|(s + n^{-1}))$. Moreover, we have*

$$\frac{\partial}{\partial z}\lambda_1(z) \ll \lambda_1(z)\left(1 + \frac{1}{z}\right) \quad \textit{and} \quad \frac{\partial^2}{\partial z^2}\lambda_1(z) \ll \lambda_1(z)\left(1 + \frac{1}{z}\right)^2.$$

Proof. The first result follows from the claim that

$$\frac{\partial \log(\lambda_i(z))}{\partial z} = O(1 + z^{-1}).$$

This follows from the above bounds on λ_i and the identity

$$\frac{\partial}{\partial z}\lambda_i(z) = -\frac{z^{-2}(\lambda_i^{k-1} + \ldots + 1)}{k\lambda_i^{k-1} - z^{-1}((k-1)\lambda_i^{k-2} + \ldots + 1)}. \tag{2.5}$$

In particular, the above allows us to check our claim for $z \gg 1$ and for $z \ll 1$. As in Lemma 2.6, the claim follows for intermediate z by a compactness argument. The bound on the second derivative follows similarly. We note that by differentiating $\lambda_1^{k+1} - \lambda_1^k - z^{-1}\left(\lambda_1^k - 1\right) = 0$ we have the identity

$$\begin{aligned}\left((k+1)\lambda_1^k - k\lambda_1^{k-1} - z^{-1}k\lambda_1^{k-1}\right)&\frac{\partial^2\lambda_1}{\partial z^2}\\ &= 2z^{-3}(\lambda_1^k - 1) - \frac{\partial\lambda_1}{\partial z}\cdot 2z^{-2}k\lambda_1^{k-1}\\ &\quad - \left(\frac{\partial\lambda_1}{\partial z}\right)^2\left((k+1)k\lambda^{k-1} - k(k-1)(1+z^{-1})\lambda_1^{k-2}\right)\end{aligned} \tag{2.6}$$

□

Lemma 2.8. *In the notation of Lemma 2.5 for $j \neq m$*

$$\left|\frac{\mu_i - \lambda_m}{\lambda_j - \lambda_m}\cdot\frac{\lambda_j}{\mu_i}\right|$$

is bounded by some constant depending only on k.

Proof. This lemma follows from Lemmas 2.6 and 2.7. In particular, in the case when neither i nor j is 1 then

$$|\lambda_j - \lambda_m| \gg |\lambda_m| \gg |\mu_i - \lambda_m|.$$

Thus $\left|\frac{\mu_i - \lambda_m}{\lambda_j - \lambda_m}\right|$ is bounded above as is $\left|\frac{\lambda_j}{\mu_i}\right|$.

If $i = 1$, the quantity in question is

$$O\left(\left|\frac{\lambda_j}{\lambda_j - \lambda_m}\right|\right) = O(1).$$

Similarly, the result follows for $j = 1$. $\square$

Proposition 2.9. *The transition matrix* $A(q^{n+1})^{-1}A(q^n) = I_k + O_k\left(s + \frac{1}{n}\right)$ *where* I_k *is the* $k \times k$ *identity matrix.*

Proof. We claim that

$$T(n,q)^{i,j} = [A(q^{n+1})^{-1}A(q^n)]_{j,i} = \prod_{m \neq i} \frac{\mu_j - \lambda_m}{\lambda_i - \lambda_m} \cdot \frac{\lambda_i}{\mu_j} = \delta_{i,j} + O(s + n^{-1}).$$

If $i \neq j$, by Lemma 2.7 the $m = j$ term of the product is

$$\frac{\mu_j - \lambda_j}{\lambda_i - \lambda_j} \cdot \frac{\lambda_i}{\mu_j} = O(s + n^{-1}) \cdot \frac{\lambda_i}{\lambda_i - \lambda_j} = O(s + n^{-1}).$$

and the remaining terms are $O(1)$ by Lemma 2.8. This proves our bound for the off-diagonal coefficients.

For $i = j$, by Lemma 2.7 each m-term in the above product equals

$$\frac{\lambda_i - \lambda_m + O(s + n^{-1})\,|\lambda_i|}{\lambda_i - \lambda_m} = 1 + O(s + n^{-1}).$$

Taking a product over m yields $1 + O(s + n^{-1})$, which proves our claim. $\square$

We conclude this section with one additional lemma dealing with the ratio of eigenvalues.

Lemma 2.10. *If* $i \neq 1$ *and* $ns \ll 1$ *then*

$$\frac{|\lambda_i(q^n)|}{|\lambda_1(q^n)|} \leq \exp\left(-c(ns)^{\frac{1}{k}}\right)$$

for some positive constant c.

Proof. This follows easily from the first case of Lemma 2.1. Namely, for $i \neq 1$

$$\frac{|\lambda_i|}{|\lambda_1|} = \exp(-\Omega(z^{\frac{1}{k}})) = \exp(-\Omega((ns)^{\frac{1}{k}})).$$

$\square$

3. Calculations of the early matrices

In this section, we construct an approximation for the vector

$$\widetilde{V}(N,q) := \left(\widetilde{v}_a^k(N,q)\right)_{a=0}^{k-1} = \prod_{n=1}^{N} m(q^n)\mathbf{e}_1$$

with $s^{-1/2} \gg N \gg s^{-\frac{1}{k+1}} \log(s^{-1})^{\frac{k}{k+1}}$.

Theorem 3.1. *Assume that $k \mid N$ for some integer N with $s^{-\frac{2}{k+2}} > N$ and N greater than a sufficiently large multiple of $s^{-\frac{1}{k+1}} \log(s^{-1})^{\frac{k}{k+1}}$, then*

$$\widetilde{v}_a^k(N,q) = (sN)^{-\frac{a}{k} - N\frac{k-1}{k}} e^{-\frac{N}{k}} \frac{1}{k^{\frac{3}{2}}} \exp\left(s^{\frac{1}{k}} N^{\frac{k+1}{k}} (k+1)^{-1} + O_k\left(sN^2 + s^{\frac{2}{k}} N^{\frac{k+2}{k}}\right)\right)$$

Before proving Theorem 3.1 we introduce some notation. Each entry of the vector is the generating function for the number of partitions with no k-sequence, no parts larger than N, and the largest missing part size is $-a \pmod k$. In this section we use the phrase "run" to refer to the gap between missing parts. Given a partition μ with parts of size at most N and no k-sequence, we let

$$\ell = \ell(\mu) = \sum_{\text{"runs"}} (k - \text{"length of run"}).$$

It is clear that $\ell \leq (k-1)N$. Note that the length of the run must be less than k and that $\ell \equiv a \pmod k$. Let $n_j = n_j(\mu)$ be the parts not appearing in μ satisfying

$$0 < n_1 < n_2 < \cdots < n_{\lfloor \frac{N+\ell}{k} \rfloor}.$$

We have

$$n_j = kj - \sum_{\text{"runs" before } n_j} (k - \text{"length of run"}).$$

We let $\{t_j\}$ be the shortenings of the runs. Namely, the length of the run before n_i is equal to

$$k - |\{j : t_j = i\}|$$

and we have

$$n_i = ki - |\{j : t_j \leq i\}|\,. \tag{3.1}$$

So we have

$$0 \leq t_1 \leq t_2 \leq \cdots \leq t_\ell \leq \left\lfloor \frac{N+\ell}{k} \right\rfloor.$$

Note that a sequence of missing parts $\{n_j\}$ determines the sequence $\{t_j\}$ and vice versa. We set

$$M := \left\lfloor \frac{N+\ell}{k} \right\rfloor = \frac{N}{k} + \frac{\ell - a}{k}.$$

So we have

$$\widetilde{v}_a^k(N,q) := \prod_{n=1}^{N} z(q^n) \cdot \sum_{\ell \equiv a \pmod k} \sum_{t_1 \leq \cdots \leq t_\ell} \prod_i z(q^{n_i})^{-1}, \tag{3.2}$$

where the sum on ℓ runs over $\ell \leq (k-1)N$. For now we ignore the term $\prod_{n=1}^{N} z(q^n)$ as this term can be dealt with separately. The idea for analyzing the remaining sum is that for N about this size runs are likely to be of size $k-1$ or $k-2$. One might interpret this as saying that all the smallest parts want to appear subject to the constraint that every kth part cannot appear. This agrees with Fristedt's probabilistic model of random partitions [11].

Next we give a lemma which says we can ignore large ℓ values.

Lemma 3.2. *In the notation above,*

$$\sum_{\substack{\ell\equiv a \pmod k\\ 2keN^{\frac{k+1}{k}}s^{\frac{1}{k}}<\ell\leq(k-1)N}} \sum_{t_1\leq\cdots\leq t_\ell}\prod_i z(q^{n_i})^{-1} = (sN)^{\frac{N}{k}}O(s^2).$$

Proof. We note that

$$\prod_i z(q^{n_i})^{-1} \leq \prod_i z(q^N)^{-1} = z(q^N)^{-\left\lfloor \frac{N+\ell}{k}\right\rfloor} \leq (sN)^{\frac{N+\ell}{k}-1}q^{O(N^2)} \leq (sN)^{\frac{N}{k}}(sN)^{\frac{\ell}{k}}s^{-1}.$$

The number of choices for t's is $\leq \binom{N+\ell-1}{\ell} \leq \binom{kN}{\ell}$. Thus

$$\sum_{t_1\leq\cdots\leq t_\ell}\prod_i z(q^{n_i})^{-1} = O\left(s^{-1}\binom{kN}{\ell}(sN)^{\frac{N}{k}}(Ns)^{\frac{\ell}{k}}\right).$$

Noting that

$$\binom{kN}{\ell} \leq \left(\frac{kNe}{\ell}\right)^{\ell},$$

this is at most

$$O\left(s^{-1}(sN)^{\frac{N}{k}}\left(keN^{\frac{k+1}{k}}s^{\frac{1}{k}}\ell^{-1}\right)^{\ell}\right) \leq O(s^{-1})(sN)^{\frac{N}{k}}2^{-\ell}.$$

We note that if N is at least a sufficiently large multiple of $s^{-\frac{1}{k+1}}\log(s^{-1})^{\frac{k}{k+1}}$, then $2^{\ell} = O(s^3)$. Summing on ℓ, yields the result. □

Proof of Theorem 3.1. We apply Lemma 3.2 to the summation in (3.2) and, unless otherwise stated, in the remainder of this proof we assume the sum on ℓ is truncated by $\ell < 2keN^{\frac{k+1}{k}}s^{\frac{1}{k}}$ at a cost of a negligible error.

We will use the following calculations throughout the proof. We have $z(q^n)^{-1} = \frac{1-q^n}{q^n}$, but $q^n = e^{-ns}$, so $1-q^n = ns\left(1+O\left(ns\right)\right)$. Moreover, $\prod q^{n_i} = e^{-\sum n_i s}$ but $s\sum n_i \leq N^2 s \ll 1$ by construction. Therefore we have

$$\prod_i z(q^{n_i})^{-1} = \prod n_i s(1+O(n_i\left|s\right|)) = s^M\prod_i n_i\cdot(1+O(sN^2)).$$

Recall that

$$n_i = ki - |\{j : t_j \leq i\}| = ki\exp\left(-\frac{|\{j:t_j\leq i\}|}{ki} + O\left(\frac{\ell\,|\{j:t_j\leq i\}|}{i^2}\right)\right). \tag{3.3}$$

So the sum becomes

$$\sum_{\ell\equiv a \pmod k} \sum_{t_1\le\cdots\le t_\ell} \prod_i z(q^{n_i})^{-1}$$

$$= \sum_{\ell\equiv a \pmod k} (sk)^M M! \sum_{t_1\le\cdots\le t_\ell} \prod_j \exp\left(-\sum_{i\ge t_j} \frac{1}{ki} + O\left(\frac{\ell}{i^2}\right)\right)\left(1+O(sN^2)\right)$$

$$= \sum_{\ell\equiv a \pmod k} (sk)^M \frac{M!}{\ell!} \sum_{t_1,\cdots,t_\ell} \exp\left(-\frac{1}{k}\sum_{j=1}^{\ell} \log\left(\frac{M}{t_j}\right) + O\left(\frac{\ell}{t_j}\right)\right)$$

$$\times \prod_j \left(1+|\{i<j : t_i = t_j\}|\right)\left(1+O(sN^2)\right)$$

$$= \sum_{\ell\equiv a \pmod k} (sk)^M \frac{M!M^\ell}{\ell!} \left(\int_0^1 t^{\frac{1}{k}} e^{O\left(\frac{\ell}{Mt}\right)} dt\right)^\ell \left(1+O\left(\frac{\ell^2}{N}+sN^2\right)\right)$$

$$= \sum_{\ell\equiv a \pmod k} (sk)^M \frac{M!M^\ell}{\ell!} \left(\int_0^1 t^{\frac{1}{k}} \left(1+O\left(\frac{\ell}{Mt}\right)\right) dt\right)^\ell \left(1+O\left(\frac{\ell^2}{N}+sN^2\right)\right)$$

$$= \sum_{\ell\equiv a \pmod k} (sk)^M \frac{M!M^\ell}{\ell!} \left(\frac{k}{k+1}\right)^\ell \left(1+O_k\left(s^{\frac{2}{k}}N^{\frac{k+2}{k}}+sN^2\right)\right),$$

where we use that $\delta \le \frac{k+1}{k}\epsilon$. The third line is obtained by removing the ordering on the t_i's. The product $\frac{1}{\ell!}\prod_j \left(1+|\{i<j : t_i=t_j\}|\right)$ accounts for the introduced over-counting. The fourth line is obtained by approximating the sum over t_j (once t_i has been fixed for $i<j$) of $t_j^{1/k}\left(1+|\{i<j : t_i=t_j\}|\right)$ by $\int t^{1/k}dt$. Additionally, in the fifth line we note that term $O\left(\frac{\ell}{Mt}\right)$ is always negative, see (3.3).

Applying Stirling's approximation to $M!$, and suppressing the errors, we see that the above is equal to

$$\left(\frac{s}{e}(N-a)\right)^{\frac{N-a}{k}} \sqrt{2\pi\frac{N-a}{k}} \sum_{\ell\equiv a \pmod k} \left(\frac{s}{e}\right)^{\frac{\ell}{k}} \left(\frac{N+\ell-a}{N-a}\right)^{\frac{N-a}{k}+\frac{1}{2}} (N+\ell-a)^{\ell\left(\frac{k+1}{k}\right)} \frac{1}{\ell!}\left(\frac{1}{k+1}\right)^\ell$$

$$= \left(\frac{s(N-a)}{e}\right)^{\frac{N-a}{k}} \sqrt{2\pi\frac{N-a}{k}} \sum_{\ell\equiv a \pmod k} \left(\frac{1}{k+1}s^{\frac{1}{k}}(N-a)^{\frac{k+1}{k}}\left(1+O\left(\frac{\ell}{N}\right)\right)\right)^\ell \frac{1}{\ell!}$$

$$= \left(\frac{s(N-a)}{e}\right)^{\frac{N-a}{k}} \sqrt{2\pi\frac{N-a}{k}} \left(\sum_{\ell\equiv a \pmod k} \left(\frac{1}{k+1}s^{\frac{1}{k}}(N-a)^{\frac{k+1}{k}}\right)^\ell \frac{1}{\ell!}\right)\left(1+O\left(s^{\frac{2}{k}}N^{\frac{k+2}{k}}\right)\right)$$

where we have used $\left(\frac{N+\ell-a}{N-a}\right)^{\frac{N-a}{k}} = \left(1+\frac{\ell}{N}\right)^{\frac{N-a}{k}} = e^{\frac{\ell}{k}}$ times a negligible error.

Extending the sum to a sum over all ℓ rather than those with $\ell < 2kes^{\frac{1}{k}}N^{\frac{k+1}{k}}$ introduces a negligible error. The completed sum over ℓ is the sum over every k-th term of an exponential.

Thus, suppressing the above error terms, we have

$$\begin{aligned}\sum_{\ell\equiv a \pmod k} &\left(s^{\frac{1}{k}}(N-a)^{\frac{k+1}{k}}(k+1)^{-1}\right)^{\ell}\frac{1}{\ell!}\\ &=\frac{1}{k}\sum_{t \pmod k}\zeta_k^{at}\exp\left(s^{\frac{1}{k}}(N-a)^{\frac{k+1}{k}}(k+1)^{-1}\zeta_k^t\right)\\ &=\frac{1}{k}\exp\left(s^{\frac{1}{k}}(N-a)^{\frac{k+1}{k}}(k+1)^{-1}\right)\left(1+O\left(\exp\left(-\frac{s^{\frac{1}{k}}N^{\frac{k+1}{k}}}{2k(k+1)}\right)\right)\right)\\ &=\frac{1}{k}\exp\left(s^{\frac{1}{k}}N^{\frac{k+1}{k}}(k+1)^{-1}\right)\left(1+O\left((sN)^{\frac{1}{k}}\right)\right)\end{aligned}$$

where we have approximated $N-a$ by N.

To finish the proof of the theorem we use

$$\prod_{n=1}^{N} z(q^n)=\prod_{n=1}^{N}(sn)^{-1}\left(1+O(ns)\right)=\frac{s^{-N}}{N!}\left(1+O\left(N^2s\right)\right)=\frac{e^N}{(sN)^N\sqrt{2\pi N}}\left(1+O\left(N^2s\right)\right).$$

□

Before concluding this section we give a comparison between $\widetilde{v}_0^k(N,q)$ and the eigenvectors of $m(q^N)$. We let $V_n^i(q)$ be the eigenvector $\begin{pmatrix} 1 & \lambda_i(q^n)^{-1} & \cdots & \lambda_i(q^n)^{-k+1}\end{pmatrix}^T$ of $m(q^n)$ corresponding to the eigenvalue $\lambda_i(q^n)z(q^n)$.

Proposition 3.3. *In the notation above, with the assumptions of Theorem 3.1 and* $V_N^i(q) = \begin{pmatrix} 1 & \lambda_i(q^N)^{-1} & \cdots & \lambda_i(q^N)^{-k+1}\end{pmatrix}^T$ *we have*

$$\begin{aligned}\widetilde{V}(N,q) = (Ns)^{-N\frac{k-1}{k}}e^{-\frac{N}{k}}\frac{1}{k^{\frac{3}{2}}}\exp\left(s^{\frac{1}{k}}N^{\frac{k+1}{k}}(k+1)^{-1}+O\left(sN^2+s^{\frac{2}{k}}N^{\frac{k+2}{k}}\right)\right)V_N^1(q)\\ +\sum_{i>1}C_N^i(q)V_N^i(q)\end{aligned}$$

where

$$C_N^i(q) \ll (Ns)^{-N\frac{k-1}{k}}e^{-\frac{N}{k}}\exp\left(s^{\frac{1}{k}}N^{\frac{k+1}{k}}(k+1)^{-1}\right)O\left(sN^2+s^{\frac{2}{k}}N^{\frac{k+2}{k}}\right).$$

Proof. Since the eigenvectors, form a basis, there exist $C_N^i(q)$ so that $\widetilde{V}(N,q)=\sum_{i\ge1}C_N^i(q)V_N^i(q)$. Applying Theorem 3.1, we have that

$$\widetilde{v}_a^k(N,q)=\widetilde{v}_0^k(N,q)(sN)^{-\frac{a}{k}}\left(1+O\left(s^{\frac{2}{k}}N^{\frac{k+2}{k}}+sN^2\right)\right).$$

By Lemma 2.1 we have that

$$\lambda_j(q^N)=e^{2\pi i\frac{(j-1)}{k}}(sN)^{\frac{1}{k}}(1+O((sN)^{\frac{2}{k}})).$$

Therefore, we have that for $0\le a\le k-1$,

$$\widetilde{v}_0^k(N,q)\left(1+O\left(s^{\frac{2}{k}}N^{\frac{k+2}{k}}+sN^2\right)\right)=\sum_{i=1}^{k}e^{-\frac{2\pi ia(j-1)}{k}}(1+O(sN)^{\frac{2}{k}})C_N^i(q).$$

In other words if B is the matrix with (a,j) entry $e^{-\frac{2\pi i a(j-1)}{k}}$, then $B+O(sN)^{\frac{2}{k}}$ times the vector of $C_N^i(q)$ equals a vector whose entries are $\widetilde{v}_0^k(N,q)\left(1+O\left(s^{\frac{2}{k}}N^{\frac{k+2}{k}}+sN^2\right)\right)$. Noting that the inverse of $B+O(sN)^{\frac{2}{k}}$ is $B^{-1}+O(sN)^{\frac{2}{k}}$ this implies that $C_N^1(q)=\widetilde{v}_0^k(N)\left(1+O\left(s^{\frac{2}{k}}N^{\frac{k+2}{k}}+sN^2\right)\right)$, and $C_N^i(q)=\widetilde{v}_0^k(N,q)O\left(s^{\frac{2}{k}}N^{\frac{k+2}{k}}+sN^2\right)$ for $i>1$. This proves our Proposition. □

Finally, the next proposition compares $\widetilde{v}_0^k(N,q)$ to the product of the eigenvalues.

Proposition 3.4. *In the notation above, with the assumptions of Theorem 3.1 we have*

$$\frac{\widetilde{v}_0^k(N,q)}{\prod_{n=1}^N \lambda_1(q^n)z(q^n)}=\frac{1}{k^{\frac{3}{2}}(2\pi)^{\frac{1-k}{2k}}}\exp\left(\frac{k-1}{2k}\log(N)+O\left(s^{\frac{2}{k}}N^{\frac{k+2}{k}}+sN^2\right)\right).$$

Proof. By Lemma 2.1 we see that the product of the first N primary eigenvalues is

$$\begin{aligned}\prod_{n=1}^N \lambda_1(q^n)z(q^n)=&\prod_{n=1}^N (ns)^{\frac{1}{k}}\left(1+\frac{1}{k}(ns)^{\frac{1}{k}}+O(ns)^{\frac{2}{k}}\right)\cdot(ns)^{-1}\left(1+O(ns)\right)\\ =&\prod_{n=1}^N (ns)^{-\frac{k-1}{k}}\left(1+\frac{1}{k}(ns)^{\frac{1}{k}}+O(ns)^{\frac{2}{k}}\right)\\ =&(N!)^{-\frac{k-1}{k}}s^{-\frac{k-1}{k}N}\exp\left(\frac{s^{\frac{1}{k}}}{k+1}N^{\frac{1+k}{k}}+O\left((sN)^{\frac{1}{k}}\right)\right)\\ =&(2\pi)^{-\frac{k-1}{2k}}(Ns)^{-N\frac{k-1}{k}}e^{N\left(1-\frac{1}{k}\right)}\\ &\times\exp\left(-\frac{k-1}{2k}\log(N)+\frac{1}{k+1}s^{\frac{1}{k}}N^{\frac{k+1}{k}}+O\left((sN)^{\frac{1}{k}}\right)\right).\end{aligned}$$

Theorem 3.1 gives the result. □

4. After the run-up

In the previous section, we computed $\widetilde{V}(N,q)=\prod_{n=1}^N m(q^n)\mathbf{e}_1$. In this section, we evaluate

$$G_k(q)=\mathbf{e}^T\prod_{n=N}^\infty m(q^n)\ \widetilde{V}(N,q)$$

We have the following proposition which shows that we only need to consider the eigenvalues and the first entry in each of the transition matrices.

Theorem 4.1. *In the notation from Lemma 2.5 for N an integer bigger than a sufficiently large multiple of $s^{-\frac{1}{k+1}}\log(s^{-1})^{\frac{k}{k+1}}$ we have*

$$G_k(q)=\prod_{n=N}^\infty \lambda_1(q^n)z(q^n)\cdot\prod_{n=N}^\infty T(n,q)^{1,1}\cdot\widetilde{v}_0^k(N,q)\cdot\left(1+O\left(s+N^{\frac{-k-1}{k}}s^{\frac{-1}{k}}\right)\right).$$

In order to prove Theorem 4.1 we will need the following lemma.

Lemma 4.2. *Let* $w(n,q) := A(q^n)^{-1}\prod_{i=1}^{n-1} m(q^i)\mathbf{e}_1$*. Then for* n *bigger than a sufficiently large multiple of* $s^{-\frac{1}{k+1}}\log(s^{-1})^{\frac{k}{k+1}}$*, we have that for* $i \neq 1$ *that*

$$|w(n,q)_i| \le O(n^{-\frac{k+1}{k}}s^{-\frac{1}{k}} + s)|w(n,q)_1|.$$

Proof of Lemma 4.2. The proof is by induction on n. Proposition 3.3 makes this result clear for n at the lowest end of the permissible range. The basic idea here is that

$$w(n+1,q) = T(n,q)D(q^n)w(n,q).$$

Now since $|\lambda_1(q^n)| > |\lambda_i(q^n)|$, multiplication by $D(q^n)$ increases the ratio of the first entry relative to the other entries. Since $T(n,q)$ is approximately I, multiplication by $T(n,q)$ does not worsen this ratio by too much.

We begin by proving our claim for $ns \ll 1$. Letting

$$u(n,q) := D(q^n)w(n,q) \tag{4.1}$$

and applying Lemma 2.1, we have that

$$\frac{|u(n,q)_i|}{|u(n,q)_1|} \le \frac{|w(n,q)_i|}{|w(n,q)_1|}(1-\Omega((ns)^{\frac{1}{k}})).$$

Next, since $T(n,q) = I_k + O(n^{-1})$, and since $|u(n,q)_i| < k|u(n,q)_1|$, we have that

$$\frac{|w(n+1,q)_i|}{|w(n+1,q)_1|} = O(n^{-1}) + \left(\frac{|w(n,q)_i|}{|w(n,q)_1|}\right)(1-\Omega((ns)^{\frac{1}{k}})).$$

Induction on n gives

$$|w(n,q)_i| \le O(n^{-\frac{k+1}{k}}s^{-\frac{1}{k}})|w(n,q)_1|$$

for all $n \ll s^{-1}$.

The argument for $ns \gg 1$ is similar. It should be noted that in this range that $\frac{|\lambda_i(q^n)|}{|\lambda_1(q^n)|}$ is bounded above by some constant less than 1 (say by $1-\epsilon$). Therefore, we have that

$$\frac{|w(n+1,q)_i|}{|w(n+1,q)_1|} = O(s) + \left(\frac{|w(n,q)_i|}{|w(n,q)_1|}\right)(1-\epsilon).$$

From this, it is easy to conclude by induction that $|w(n,q)_i| = O(s)|w(n,q)_1|$. □

Remark 4.3. *It should be noted that the bound in Lemma 4.2 is not tight for small* n *(a stronger bound is given in Proposition 3.3). The bound of* $n^{-\frac{k+1}{k}}s^{-\frac{1}{k}}$ *would be tight given our analysis if all we use is that* $T(n,q)^{1,i} = O(n^{-1})$ *and that* $\left|\frac{\lambda_i(q^n)}{\lambda_1(q^n)}\right| = 1-\Omega((ns)^{\frac{1}{k}})$*. In order to obtain a tighter analysis, one can note that the* $T(n,q)^{1,j}$ *are roughly constant in* n *and that* $\frac{\lambda_i}{\lambda_1}$ *is roughly* ω^i*, where* ω *is a primitive* k*th root of unity. By our previous analysis,* $\frac{w_i(n+1,q)}{w_1(n+1,q)}$ *is approximately* $\frac{\lambda_i(q^n)}{\lambda_1(q^n)}\left(T(n,q)^{1,i} + \left(\frac{w_i(n,q)}{w_1(n,q)}\right)\right)$*. Approximating each* $\frac{\lambda_i}{\lambda_1}$ *by* $\omega^i(1-(ns)^{\frac{1}{k}})$ *and each* $T(n,q)^{1,i}$ *by a constant of order* n^{-1}*, we note that resulting recurrence leads to terms of size* $O(n^{-1})$ *due to cancelation that is not captured in our analysis.*

We are now prepared to prove Theorem 4.1.

Proof of Theorem 4.1. We claim that

$$w(n+1,q)_1 = w(n,q)_1\lambda_1(q^n)z(q^n)T(n,q)^{1,1}(1+O(\min(n^{-\frac{2k+1}{k}}s^{-\frac{1}{k}}, s^2z(q^n)))).$$

Or equivalently (since $u(n,q)_1 = \lambda_1(q^n)z(q^n)w(n,q)_1$) that

$$w(n+1,q)_1 = u(n,q)_1T(n,q)^{1,1}(1+O(\min(n^{-\frac{2k+1}{k}}s^{-\frac{1}{k}}, s^2z(q^n)))).$$

It is clear that

$$w(n+1,q)_1 = \sum_j T(n,q)^{1,j}u(n,q)_j$$

Hence we need to show

$$\max_{j\neq 1}\left(T(n,q)^{1,j}\cdot\frac{|u(n,q)_j|}{|u(n,q)_1|}\right) = O(\min(n^{-\frac{2k+1}{k}}s^{-\frac{1}{k}}+s, s^2z(q^n))).$$

If $ns \ll 1$, this follows since $T(n,q)^{1,j} \ll n^{-1}$, and $\frac{|u(n,q)_j|}{|u(n,q)_1|} \leq \frac{|w(n,q)_j|}{|w(n,q)_1|} = O(n^{-\frac{k+1}{k}}s^{-1})$. Otherwise, this follows from noting that $T(n,q)^{1,j} \ll s$ and

$$\frac{|u(n,q)_j|}{|u(n,q)_1|} = \left(\frac{|\lambda_j(q^n)|}{|\lambda_1(q^n)|}\right)\left(\frac{|w(n,q)_j|}{|w(n,q)_1|}\right) = O(z(q^n)s).$$

This proves the claim.

Therefore we have that

$$\lim_{n\to\infty} w(n,q)_1 = \prod_{n=N+1}^{\infty}\lambda_1(q^n)z(q^n)T(n,q)^{1,1}\cdot\exp\left(O\left(\sum_{n=N+1}^{\infty}\min(n^{-\frac{2k+1}{k}}s^{-\frac{1}{k}}, s^2z(q^n))\right)\right).$$

The sum in the error term is at most

$$\sum_{n=N+1}^{\lfloor s^{-1}\rfloor} n^{-\frac{2k+1}{k}}s^{-\frac{1}{k}} + \sum_{n=\lfloor s^{-1}\rfloor}^{\infty} s^2z(q^n).$$

The first term is $O\left(N^{-\frac{k+1}{k}}s^{-\frac{1}{k}}\right)$ and the latter term is $O\left(s^2\sum_{n=1}^{\infty}e^{-ns}\right) = O(s)$. □

The following theorem is enough to deduce Theorem 1.4 and thus Theorem 1.2

Theorem 4.4. *With N as above we have*

$$\prod_{n=N}^{\infty} T(n,q)^{1,1} = k^{\frac{1}{2}}\exp\left(-\frac{k-1}{2k}\log(Ns) + O\left((Ns)^{\frac{1}{k}} + N^{-1} + s\right)\right).$$

Proof. Throughout this proof we use the notation of Lemma 2.5 and often suppress the dependence on n. We have

$$T(n-1,q)^{1,1} = \prod_{m\neq 1}\frac{\mu_1-\lambda_m}{\lambda_1-\lambda_m}\cdot\frac{\lambda_1}{\mu_1}$$

and

$$\mu_1(q^n) = \lambda_1(q^{n-1}) = \lambda_1(q^n) - \lambda_1'(q^n) + O(\lambda_1''(q^n))$$

where $\lambda_1'(q^n) = \frac{\partial}{\partial n}\lambda_1(q^n)$. Therefore,

$$\frac{\mu_1 - \lambda_m}{\lambda_1 - \lambda_m} \cdot \frac{\lambda_1}{\mu_1} = 1 + \lambda_1' \left(\frac{1}{\lambda_1 - \lambda_m} - \frac{1}{\lambda_1} \right) + O_k \left(\left(\frac{\lambda_1''}{\lambda_1} + \left(\frac{\lambda_1'}{\lambda_1} \right)^2 \right) \right)$$

Hence,

$$T(n-1,q)^{1,1} = \exp\left(-\lambda_1' \sum_{m \neq 1} \left(\frac{1}{\lambda_1 - \lambda_m} - \frac{1}{\lambda_1} \right) + O_k \left(\left(\frac{\lambda_1''}{\lambda_1} + \left(\frac{\lambda_1'}{\lambda_1} \right)^2 \right) \right) \right).$$

To estimate the big-O term for $ns \ll 1$ we use (2.5) and (2.6) and Lemma 2.1 to obtain

$$\begin{aligned}
\frac{1}{\lambda_1}\frac{\partial \lambda_1}{\partial n} &= -\, s\frac{1}{\lambda_1}\frac{\partial \lambda_1}{\partial z} \cdot z(q^n)^2 e^{ns} = O\left(\frac{1}{n}\right) \\
\frac{1}{\lambda_1}\frac{\partial^2 \lambda_1}{\partial n^2} &= \frac{s^2 e^{2ns}}{\lambda_1} \left(\frac{\partial^2 \lambda_1}{\partial z^2} \cdot z(q^n)^4 + \frac{\partial \lambda_1}{\partial z} \cdot z(q^n)^3 \right) = O\left(\frac{1}{n^2}\right).
\end{aligned}$$

For $ns \gg 1$ we use Lemma 2.7 to obtain

$$\frac{1}{\lambda_1}\frac{\partial \lambda_1}{\partial n} = O\left(se^{-ns}\right) \quad \text{and} \quad \frac{1}{\lambda_1}\frac{\partial^2 \lambda_1}{\partial n^2} = O\left(s^2 e^{-ns}\right).$$

Therefore

$$\begin{aligned}
\prod_{n=N}^{\infty} T(n-1,q)^{1,1} \exp\left(\lambda_1' \sum_{m \neq 1} \left(\frac{1}{\lambda_1 - \lambda_m} - \frac{1}{\lambda_1} \right) \right) &= \exp\left(\sum_{n=N}^{\lfloor \frac{1}{s} \rfloor} O\left(\frac{1}{n^2}\right) + O\left(s^2 \sum_{n=\lfloor \frac{1}{s} \rfloor}^{\infty} e^{-ns} \right) \right) \\
&= \exp\left(O\left(\frac{1}{N} + s \right) \right).
\end{aligned}$$

Let $P(\lambda, z) := \lambda^k - z^{-1}\left(\lambda^{k-1} + \cdots + \lambda + 1\right)$. We have

$$2\sum_{m \neq 1} \frac{1}{\lambda_1 - \lambda_m} = \frac{\frac{\partial^2}{\partial \lambda^2} P(\lambda, z)}{\frac{\partial}{\partial \lambda} P(\lambda, z)}\Big|_{\substack{\lambda=\lambda_1 \\ z=z(N,q)}} =: R_k(\lambda_1(q^n)). \tag{4.2}$$

Therefore,

$$\begin{aligned}
\prod_{n=N}^{\infty} T(n,q)^{1,1} &= \prod_{n=N}^{\infty} T(n-1,q)^{1,1}(1 + O(N^{-1} + s)) \\
&= \exp\left(-\sum_{n=N}^{\infty} \left(\frac{1}{2}\lambda_1'(q^n) R_k(\lambda_1(q^n)) - (k-1)\frac{\lambda_1'(q^n)}{\lambda_1(q^n)} \right) + O\left(N^{-1} + s\right) \right)
\end{aligned}$$

We apply Euler-MacLaurin to approximate the sum by an integral. The error from the terms $\frac{\lambda_1'(q^n)}{\lambda_1(q^n)}$ introduces an error of size

$$\int_N^{\infty} \left(\frac{\lambda_1''(q^n)}{\lambda_1(q^n)} + \left(\frac{\lambda_1'(q^n)}{\lambda_1(q^n)} \right)^2 \right) dn = O\left(N^{-1} + s\right)$$

as above. Thus, we have

$$\prod_{n=N}^{\infty} T(n,q)^{1,1} = \exp\left(-\int_N^{\infty}\left(\frac{1}{2}\lambda_1'(q^x)R_k(\lambda_1(q^x)) - (k-1)\frac{\lambda_1'(q^x)}{\lambda_1(q^x)}\right)dx + O\left(N^{-1}+s\right)\right)$$
$$= \exp\left(-\int_{\lambda_1(q^N)}^{\infty}\frac{R_k(x)}{2} - \frac{k-1}{x}dx + O\left(N^{-1}+s\right)\right)$$

In order to evaluate the integral $\int R_k(x)dx$, we let $a(\lambda) = \lambda^k$ and $b(\lambda) = \lambda^{k-1} + \cdots + 1$. We then have that $z^{-1} = \frac{a(\lambda_1)}{b(\lambda_1)}$. Therefore,

$$R_k(\lambda) = \frac{a''(\lambda) - z^{-1}b''(\lambda)}{a'(\lambda) - z^{-1}b'(\lambda)} = \frac{a''(\lambda)b(\lambda) - a(\lambda)b''(\lambda)}{a'(\lambda)b(\lambda) - a(\lambda)b'(\lambda)} = \frac{\partial}{\partial\lambda}\log(a'(\lambda)b(\lambda) - a(\lambda)b'(\lambda)).$$

Letting

$$\begin{aligned} Q(\lambda) &:= a'(\lambda)b(\lambda) - a(\lambda)b'(\lambda) \\ &= k\lambda^{k-1}(\lambda^{k-1} + \cdots + 1) - \lambda^k((k-1)\lambda^{k-2} + \cdots + 1) \\ &= \lambda^{2k-2} + 2\lambda^{2k-3} + \cdots + k\lambda^{k-1}, \end{aligned}$$

we have that

$$\int_{\lambda_1(q^N)}^{\infty}\frac{R_k(x)}{2} - \frac{k-1}{x}dx = \frac{1}{2}\left[\log\left(Q(\lambda)\lambda^{-2k+2}\right)\right]_{\lambda_1(N)}^{\infty}.$$

We note that for $\lambda \gg 1$ that $Q(\lambda)\lambda^{-2k+2} = 1 + O(\lambda^{-1})$, and therefore,

$$\lim_{\lambda\to\infty}\log\left(Q(\lambda)\lambda^{-2k+2}\right) = 0.$$

For $\lambda \ll 1$, we have that $Q(\lambda)\lambda^{-2k+2} = k\lambda^{-k+1}(1 + O(\lambda))$. Therefore

$$\prod_{n=N}^{\infty} T(n,q)^{1,1} = \exp\left(\frac{1}{2}\log\left(k\lambda_1(q^N)^{-k+1}(1 + O(\lambda_1(q^N)))\right) + O\left(N^{-1}+s\right)\right).$$

By Lemma 2.1, we have

$$\prod_{n=N}^{\infty} T(n,q)^{1,1} = \exp\left(-\frac{k-1}{2}\log\left(\lambda_1(q^N)\right) + \frac{1}{2}\log(k) + O\left((Ns)^{\frac{1}{k}} + N^{-1} + s\right)\right)$$
$$= \exp\left(-\frac{k-1}{2k}\log(Ns) + \frac{1}{2}\log(k) - \frac{k-1}{2k}(Ns)^{\frac{1}{k}} + O\left((Ns)^{\frac{1}{k}} + N^{-1} + s\right)\right).$$

□

In the next section, we analyze the product of the primary eigenvalues.

5. The Product of the Primary Eigenvalues

In this section, we estimate

$$\prod_{n=1}^{\infty}\lambda_1(q^n)z(q^n) = \exp\left(\sum_{n=1}^{\infty}\log(\lambda_1(q^n)) + \log\left(\frac{q^n}{1-q^n}\right)\right).$$

Theorem 5.1. *In the notation above we have*

$$\sum_{n=1}^{\infty}\log\left(\lambda_1(q^n)z(q^n)\right)=\frac{\pi^2}{6s}\left(1-\frac{2}{k(k+1)}\right)+\left(\frac{k-1}{2k}\right)\log(s)-\left(\frac{k-1}{2k}\right)\log(2\pi)+O_k\left(s^{\frac{1}{k}}\right).$$

We start with the following lemma which closely resembles Euler-MacLaurin summation.

Lemma 5.2. *For suitable functions h and $n\geq 1$ we have*

$$\begin{aligned}h(n)&=\int_{n-\frac{1}{2}}^{n+\frac{1}{2}}h(z)dz-\int_{n-\frac{1}{2}}^{n+\frac{1}{2}}h'(x)\left([x]-x+\frac{1}{2}\right)dx\\&=\int_{n-\frac{1}{2}}^{n+\frac{1}{2}}h(z)dz-\frac{1}{2}\int_{n-\frac{1}{2}}^{n+\frac{1}{2}}h''(x)\left([x]-x+\frac{1}{2}\right)^2dx\end{aligned}$$

where $[x]$ denotes the integer part of x.

Proof. To see this note that for any function $h(z)$ we have

$$h(z)=h(n)+h'(n)(z-n)+\int_n^z h''(x)(z-x)dx.$$

Integrating from $n-\frac{1}{2}$ to $n+\frac{1}{2}$ gives the second result. Integration by parts on each interval $[n,n+\frac{1}{2}]$ and $[n-1/2,n]$ gives the first result. □

Define the function $f_k(e^{-x})$ to be the increasing function satisfying

$$f_k(e^{-x})^{k+1}-f_k(e^{-x})^k=e^{-x(k+1)}-e^{-xk}. \tag{5.1}$$

Since $\lambda_1(q^n)^k = z(q^n)^{-1}\left(\lambda_1(q^n)^{k-1}+\cdots\lambda_1(q^n)+1\right)$, multiplying by $\lambda_1(q^n)-1$ we have $\lambda_1(q^n)^{k+1}-\lambda_1(q^n)^k=z(q^n)^{-1}(\lambda_1(q^n)^k-1)=q^{-n}\lambda_1^k-q^{-n}-\lambda_1^k+1$. Therefore $f_k(e^{-ns})=\lambda_1(q^n)q^n$.

Remark 5.3. *This function $f_k(e^{-x})$, and certain generalizations, are studied in* [15].

Proof of Theorem 5.1. The modularity of the Dedekind η-function gives

$$\sum_{n=1}^{\infty}\log\left(1-q^n\right)=\frac{\pi^2}{6s}+\frac{1}{2}\log(s)-\frac{1}{2}\log(2\pi)-\frac{s}{24}+O(s^M) \tag{5.2}$$

for any $M>0$. Additionally, by Lemma 5.2, we have

$$\sum_{n=1}^{\infty}\log\left(1-q^n\right)=\int_0^{\infty}\log(1-e^{-xs})dx-\int_0^{\frac{1}{2}}\log(1-e^{-xs})dx-s\int_{\frac{1}{2}}^{\infty}\frac{e^{-xs}}{1-e^{-xs}}\left([x]-x+\frac{1}{2}\right)dx.$$

Noting that $\int_0^\infty\log(1-e^{-xs})dx=\frac{\pi^2}{6s}$ and $\int_0^{\frac{1}{2}}\log(1-e^{-xs})dx=\frac{1}{2}\log(s)+\int_0^{\frac{1}{2}}\log(x)dx+O(s)$ we may conclude that

$$-\int_0^{\frac{1}{2}}\log(x)dx-s\int_{\frac{1}{2}}^{\infty}\frac{e^{-xs}}{1-e^{-xs}}\left([x]-x+\frac{1}{2}\right)dx=\frac{1}{2}\log(2\pi)+O(s). \tag{5.3}$$

Following the notation of Section 3 of [8] we define

$$g_k(xs)=-\log(f_k(e^{-xs})). \tag{5.4}$$

By Lemma 5.2,

$$\sum_{n=1}^{\infty} g_k(ns) = \int_0^{\infty} g_k(xs)dx - \int_0^{\frac{1}{2}} g_k(xs)dx - s\int_{\frac{1}{2}}^{\infty} g_k'(xs)\left([x]-x+\frac{1}{2}\right)dx. \tag{5.5}$$

Theorem 1 of [15] gives $\int_0^\infty g_k(xs)dx = \frac{1}{s}\frac{\pi^2}{3k(k+1)}$. Lemma 2.1 gives that for $sx \ll 1$

$$g_k(xs) = -\log\left(f_k(e^{-xs})\right) = -\frac{1}{k}\log(xs) + \frac{1}{k}(xs)^{\frac{1}{k}} + O\left((xs)^{\frac{2}{k}}\right)$$

Therefore, we have

$$-\int_0^{\frac{1}{2}} g_k(xs)dx = \frac{1}{2k}\log(s) - \frac{1}{k}\int_0^{\frac{1}{2}}\log(x)dx + O\left(s^{\frac{1}{k}}\right). \tag{5.6}$$

Let $M = \lfloor s^{-\frac{1}{k}} \rfloor$. Then we have

$$\begin{aligned} s\int_{M+\frac{1}{2}}^{\infty} g_k'(xs)\left([x]-x+\frac{1}{2}\right)dx &= \frac{s^2}{2}\int_{M+\frac{1}{2}}^{\infty} g_k''(xs)\left([x]-x+\frac{1}{2}\right)^2 dx \\ &\ll s\int_{Ms}^{\infty} g_k''(w)dw \ll M^{-1} \ll s^{\frac{1}{k}} \end{aligned} \tag{5.7}$$

where we use $g'(Ms) = O_k\left(\frac{1}{Ms}\right)$ (see, for instance, Lemma 3.1 of [8]).

To estimate the integral of g_k' from $\frac{1}{2}$ to $M+\frac{1}{2}$ we take the logarithmic derivative of $f_k(e^{-w})^{k+1} - f_k(e^{-w})^k = e^{-w(k+1)} - e^{-wk}$ to obtain

$$g_k'(w) = 1 - \frac{1}{k}\frac{e^{-w}}{e^{-w}-1} + \frac{1}{k}e^{-w}\frac{f_k'(e^{-w})}{1-f_k(e^{-w})}.$$

Therefore

$$\begin{aligned} s\int_{\frac{1}{2}}^{M+\frac{1}{2}} g_k'(xs)\left([x]-x+\frac{1}{2}\right)dx = &-\frac{s}{k}\int_{\frac{1}{2}}^{M+\frac{1}{2}} \frac{e^{-xs}}{1-e^{-xs}}\left([x]-x+\frac{1}{2}\right)dx \\ &+\frac{s}{k}\int_{\frac{1}{2}}^{M+\frac{1}{2}} e^{-xs}\frac{f_k'(e^{-xs})}{1-f_k(e^{-xs})}\left([x]-x+\frac{1}{2}\right)dx \end{aligned} \tag{5.8}$$

Observe that we have

$$\begin{aligned} &\int_{\frac{1}{2}}^{M+\frac{1}{2}} \frac{e^{-xs}}{1-e^{-xs}}\left([x]-x+\frac{1}{2}\right)dx \\ &= \int_{\frac{1}{2}}^{\infty} \frac{e^{-xs}}{1-e^{-xs}}\left([x]-x+\frac{1}{2}\right)dx - \int_{M+\frac{1}{2}}^{\infty} \frac{e^{-xs}}{1-e^{-xs}}\left([x]-x+\frac{1}{2}\right)dx \\ &= \int_{\frac{1}{2}}^{\infty} \frac{e^{-xs}}{1-e^{-xs}}\left([x]-x+\frac{1}{2}\right)dx + \frac{s}{2}\int_{M+\frac{1}{2}}^{\infty} \frac{e^{-xs}}{(1-e^{-xs})^2}\left([x]-x+\frac{1}{2}\right)^2 dx \\ &= \int_{\frac{1}{2}}^{\infty} \frac{e^{-xs}}{1-e^{-xs}}\left([x]-x+\frac{1}{2}\right)dx + O(se^{-Ms}). \end{aligned} \tag{5.9}$$

Additionally, integrating by parts we obtain

$$\begin{aligned} &s\int_{\frac{1}{2}}^{M+\frac{1}{2}} e^{-xs}\frac{f_k'(e^{-xs})}{1-f_k(e^{-xs})}\left([x]-x+\frac{1}{2}\right)dx \\ &\ll s\cdot \frac{e^{-xs}f_k'(e^{-xs})}{1-f_k(e^{-xs})}\Big|_{\frac{1}{2}}^{M+\frac{1}{2}} \ll_k s^{\frac{1}{k}}\left(1+M^{-\frac{k-1}{k}}\right) \end{aligned} \tag{5.10}$$

where we have used that monotonicity of $\log(1-f_k(w))$ and $f_k'(z)=O\left(z^{\frac{1-k}{k}}\right)$ for z near 0. Returning to (5.5) and using (5.3) and (5.7)-(5.10)

$$\begin{aligned} &-\frac{1}{k}\int_0^{\frac{1}{2}}\log(x)dx - s\int_{\frac{1}{2}}^{\infty} g_k'(xs)\left([x]-x+\frac{1}{2}\right)dx \\ &=\frac{1}{k}\left(-\int_0^{\frac{1}{2}}\log(x)dx - s\int_{\frac{1}{2}}^{\infty}\frac{e^{-xs}}{1-e^{-xs}}\left([x]-x+\frac{1}{2}\right)dx\right)+O\left(s^{\frac{1}{k}}+M^{-1}\right) \\ &=\frac{1}{2k}\log(2\pi)+O(s^{\frac{1}{k}}) \end{aligned}$$

Finally, this together with (5.5) and (5.6) gives the result. □

6. Proof of Theorem 1.4

In this section, we prove Theorem 1.4 and thus Theorem 1.2.

Proof of Theorem 1.4. We have $G_k(q)=\mathbf{e}^T\prod_{n=N+1}^{\infty}m(q^n)\cdot\prod_{n=1}^{N}m(q^n)\mathbf{e}_1$. It follows from Theorem 4.1, Proposition 3.4, and Theorems 4.4 and 5.1 that for appropriate N,

$$G_k(e^{-s})=\frac{1}{k}\exp\left(\frac{\pi^2}{6s}\left(1-\frac{2}{k(k+1)}\right)+O\left(N^{-\frac{k+1}{k}}s^{-\frac{1}{k}}+sN^2+s^{\frac{2}{k}}N^{\frac{k+2}{k}}+N^{-1}\right)\right).$$

Setting $N=\left\lfloor s^{-\frac{3}{2k+3}}\right\rfloor$ yields the result. □

7. Proof of Theorem 1.8

In this section we apply a result of Ingham [16] to deduce the asymptotics for $p_k(n)$ from the asymptotics of $G_k(q)$ as $q\to 1$. In particular, we have the following result which is a special case of Theorem 1 of [16] and is given as Theorem 4.1 of [10].

Theorem 7.1 (Ingham). *Let $f(z)=\sum_{n=0}^{\infty}a(n)z^n$ be a power series with real nonnegative coefficients and radius of convergence equal to 1. If there exists $A>0$, λ, $\alpha\in\mathbb{R}$ such that*

$$f(z)\sim\lambda(-\log(z))^{\alpha}\exp\left(-\frac{A}{\log(z)}\right)$$

as $z\to 1^-$, then

$$\sum_{m=0}^{n}a(m)\sim\frac{\lambda}{2\sqrt{\pi}}\frac{A^{\frac{\alpha}{2}-\frac{1}{4}}}{n^{\frac{\alpha}{2}+\frac{1}{4}}}\exp\left(2\sqrt{An}\right)$$

as $n\to\infty$.

Proof of Theorem 1.8. By Lemma 10 of [15] $(1-q)G_k(q) = \sum_{n=0}^{\infty}(p_k(n) - p_k(n-1))q^n$ has nonnegative coefficients. Applying Theorems 1.4 and 7.1 gives the result. □

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