Math 96:
Polynomials Techniques

October 31st, 2024

1 Introduction

A polynomial in one variable is a function of the form $f(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$ for some constants $a_0, a_1, \ldots, a_n$. More generally, a polynomial in several variables is a function $f(x_1, \ldots, x_k)$ that is a sum of monomials, that is functions of the form $cx_1^{e_1}x_2^{e_2}\cdots x_n^{e_n}$ for c some constant and e_i non-negative integers. These functions show up in many places and are popular both as topics of questions and tools that can be used in problem solving.

2 Factorization and Roots

Like with integers, polynomials exhibit a form of unique factorization. In particular, if you have a polynomial f whose coefficients lie in $\mathbb{Z}$ or $\mathbb{Q}$ or $\mathbb{R}$ or $\mathbb{C}$ (or any other field or unique factorization domain) in any number of variables, then f can always be written as a product of *irreducibles*, that is polynomials that cannot be written as a product of two non-constant polynomials, with coefficients in the same ring as f. Furthermore, this factorization is unique up to reordering the factors and multiplying the factors by constants.

This factorization is particularly interesting for univariate polynomials. A polynomial $f(x)$ has a root at r if $f(r) = 0$. It is not hard to show that this happens if and only if $(x-r)$ is a factor of f. This implies for example that if $f(x)$ is of degree-d (i.e. the largest power of x with a non-zero coefficient is d), then f can have at most d roots (since it can admit at most d linear factors). This is particularly interesting over $\mathbb{C}$, where the *Fundamental Theorem of Algebra* says that every polynomial f factors into linear factors. Furthermore, over $\mathbb{R}$, every polynomial factors into linear and quadratic factors.

1940 A6: $f(x)$ is a polynomial [with real coefficients] of degree n, such that a power of $f(x)$ is divisible by a power of its derivative $f'(x)$; i.e. $[f(x)]^p$ is divisible by $[f'(x)]^q$; p, q positive integers. Prove that $f(x)$ is divisible by $f'(x)$ and that $f(x)$ has a single root of multiplicity n.

3 Coefficients and Elementary Symmetric Polynomials

So if we know that a polynomial $f(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_0$ over the complex numbers can be factored as $(x - r_1)(x - r_2)\cdots(x - r_n)$, what exactly is the relationship between the coefficients a_i and the roots r_i? On the one hand, the r_i are the roots of the polynomial $f(x)$ (i.e. the values r so that $f(r) = 0$) counted with multiplicity. On the other hand, expanding out $(x-r_1)(x-r_2)\cdots(x-r_n)$, we find that a_k is $(-1)^k$ times the *kth elementary symmetric polynomial in the r_i's*, that is the sum over all products of k of the r_i terms. Of particular note are a_1 and a_n with $a_1 = -\sum r_i$ and $a_n = (-1)^n \prod r_i$.

1940 B5: Determine all rational values for which a, b, c are the roots of

$$x^3 + ax^2 + bx + c = 0.$$

4 Polynomials, Roots, and Computations

There are a number of tricks that you can play if you know polynomial identities. If you know that $f(x) = g(x)$ as polynomials, then plugging in any value of x will give you a potentially useful equality. This often occurs when one can identify the factorization of a polynomial with its expanded form, often involving the roots of unity. In particular, the polynomial $f(x) = x^n - 1$ has roots of the form $e^{2\pi im/n}$ for integer m. We can thus factor it is

$$f(x) = x^n - 1 = \prod_{m=1}^{n} (x - e^{2\pi im/n}).$$

1955 A2: $A_1A_2\ldots A_n$ is a regular polygon inscribed in a circle of radius r and center O. P is a point on the line OA, extended beyond A_1. Show that

$$\prod_{i=1}^{n} \overline{PA_i} = \overline{OP}^n - r^n.$$

5 Finite Differences

Let $f(x) = a_0 + a_1x + a_2x^2 + \ldots + a_dx^d$ be a degree-d univariate polynomial. Consider the finite difference $f(x + n) - f(x)$ for some non-zero number n. By the Binomial Theorem this is

$$\sum_{i=0}^{d} a_i \sum_{j=0}^{i} x^j n^{i-j} \binom{i}{j} - \sum_{i=0}^{d} a_i x^i = \sum_{i=0}^{d} a_i \sum_{j=0}^{i-1} x^j n^{i-j} \binom{i}{j}.$$

In particular, note that the highest degree, x^d terms here cancel out, leading to a polynomial with degree $d - 1$, and leading coefficient dna_d. It is also possible

(though somewhat more cumbersome) to track the next few highest coefficients. This is a very useful thing to know, especially when one repeatedly takes such finite differences until the resulting polynomial becomes a constant.

2020 B1: For a positive integer n, define $d(n)$ to be the sum of the digits of n when written in binary (for example, $d(13) = 1 + 1 + 0 + 1 = 3$). Let

$$S = \sum_{k=1}^{2020} (-1)^{d(k)} k^3.$$

Determine S modulo 2020.