Math 96:
Combinatorics Techniques

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1 Introduction

Combinatorics is the study of counting things. While this might sound trivial, there are a lot of situations where the objects being considered are too abstract or too large to count directly and so more complicated techniques are required. Combinatorics in general also covers a grab bag of other results about discrete objects.

2 Basic Counting Techniques

The most basic ways to count things are to break the counting problem you have into simpler parts.

The addition rule says that if you can split the thing you are trying to count into disjoint cases, then you can take the sum of the number of things in the individual cases. In particular, if A and B are disjoint sets, then $|A \cup B| = |A| + |B|$.

The multiplication rule says that if an object is constructed by making a pair of choices the total number of ways to make both choices is the number of ways to make the first choice times the number of ways to make the second. In particular, you have $|A \times B| = |A||B|$. More generally, if you make the two choices sequentially, this still holds even if the exact options available for the second choice depend on your first choice so long as the number of possibilities doesn't change.

These rules can often be repeated many times and applied in careful order in order to count more complicated objects. Furthermore, being able to accurately count things will often have surprising side benefits.

1985 A1: Determine, with proof, the number of ordered triples (A_1, A_2, A_3) of sets which have the property that

(i) $A_1 \cup A_2 \cup A_3 = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, and

(ii) $A_1 \cap A_2 \cap A_3 = \emptyset$.

Express your answer in the form $2^a 3^b 5^c 7^d$, where a, b, c, d are nonnegative integers.

3 Binomial Coefficients

One slightly less obvious counting formula is that for binomial coefficients. The binomial coefficient $\binom{n}{k}$ ("n choose k") counts the number of ways to select a subset of size k from a specific set of size n. There are several important facts to know about these coefficients:

$$\begin{aligned}
\binom{n}{k} &= \frac{n!}{k!(n-k)!} \\
\binom{n}{k} &= \binom{n}{n-k} \\
\binom{n+1}{k} &= \binom{n}{k} + \binom{n}{k-1} \\
(x+y)^n &= \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.
\end{aligned}$$

One nice application of binomial coefficients is the "sticks and stones" approach for counting compositions. In particular, if you want to count the number of sequences of non-negative integers $a_1, a_2, \ldots, a_k$ with $a_1 + a_2 + \ldots + a_k = m$ for some fixed k and m you can do this as follows: For any such sequence (say $3, 1, 4$) draw a sequence of dots and lines where each a_i is written as that many dots with lines between the pairs (so our example corresponds to $\circ\circ\circ|\circ|\circ\circ\circ$). We note that this leads to a total of m dots and $k-1$ lines with the lines going between different pairs of dots. If we think of the m dots as being in fixed positions, we need to pick $k-1$ of the $m-1$ spaces between dots to contain lines. The total number of ways to do this is $\binom{m-1}{k-1}$. Similar formulas can be derived if the a_i's are taken to be non-negative, if they sum to at most m or if k can be any number.

2005 B4: For positive integers m and n, let $f(m, n)$ denote the number of n-tuples $(x_1, x_2, \ldots, x_n)$ of integers such that $|x_1| + |x_2| + \ldots + |x_n| \leq m$. Show that $f(m, n) = f(n, m)$.

4 Inclusion-Exclusion

Often when trying to count something, the things we are trying to count can be split up into several different cases. If these cases are disjoint, we can simply add up the number of things from each case. However, if the cases are not disjoint, this may lead to overcounting. For example, if we wanted to count the number of students in class with either black hair or blond hair, we could add up the number of students with black hair and the number of students with blond hair.

However, if we wanted to count the number of students with black hair or green eyes, then if we simply added up the number of students with black hair and the number with green eyes, we would have counted any students with both black hair and green eyes twice. We can deal with this problem by figuring out the number of students we are overcounting and correcting for it. In this case, if we then subtracted off the number of students with black hair and green eyes, we would be good.

More abstractly, we have two sets A and B and want to count the size of $A \cup B$. If A and B are disjoint, we can take the sum $|A \cup B| = |A| + |B|$. However, if they are not disjoint, we need a correction term $|A \cup B| = |A| + |B| - |A \cap B|$. More generally, if we have n sets $A_1, A_2, \ldots, A_n$ and want to count the number of elements in any of them (i.e. the size of their union), we can do this using what is known as Inclusion-Exclusion. We compute it by first adding up the sizes of the individual sets. We then must subtract off the sizes of the intersections of each pair. Then we need to add back in the sizes of intersections of triples, and subtract off quadruples and so forth. In general, we get the formula,

$$|A_1 \cup A_2 \cup \ldots \cup A_n| = \sum_{k=1}^{n} (-1)^{k+1} \sum_{1 \leq i_1 < i_2 < \ldots < i_k \leq n} |A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_k}|.$$

2015 A5: Let q be an odd positive integer, and let N_q denote the number of integers a such that $0 < a < q/4$ and $\gcd(a, q) = 1$. Show that N_q is odd if and only if q is of the form p^k with k a positive integer and p a prime congruent to 5 or 7 modulo 8.

5 Counting Things in Two Different Ways and Linearity of Expectation

One of the most useful tools in combinatorics is counting the same object in two different ways. This gives two formulas for the same number, and thus can be used to prove a number of useful equations. In fact, essentially all of the facts about binomial coefficients above can be proved this way.

Another version of this is what is known as *linearity of expectation*. This says that if you have some quantity given as a $X = A + B$ then the average of X is the same as the average of A plus the average of B. So for example, if a students grade in a class in based on their total score over several tests, the average total score in the class is the sum of the average scores on each of the tests. This is a very versatile technique, especially when you write things in terms of indicator functions (that is a quantity that is 1 if some event happens and 0 if it does not).

1965 B2: In a round-robin tournament with n players $P_1, P_2, \ldots, P_n$ (where $n > 1$), each player plays one game with each of the other players and the rules are such that no ties can occur. Let w_r and l_r be the number of games won and

lost, respectively, by $P - r$. Show that

$$\sum_{r=1}^{n} w_r^2 = \sum_{r=1}^{n} l_r^2.$$