Math 96:
Calculus Techniques

November 14th, 2025

1 Introduction

Calculus problems are fairly common on the Putnam. While you all should have had at least some exposure to calculus already, there are some tricks and ideas that are worth keeping in mind.

2 Convergent Sums and Integrals

One important topic is that of showing convergence of sums. One of the most common ways for this to happen is for the sum in question to be absolutely convergent, that is for the sum of the absolute values of the terms of the sum to be convergent. If as sum is absolutely convergent, it and any rearrangement of its terms will be convergent and will always sum to the same value. Showing absolute convergence is usually about estimating how quickly the terms go to 0. If the terms of the sum are decreasing, then the power test says that it will converge if the n^{th} term is much smaller than $1/(n^{1+c})$ for any positive c, but will diverge if the terms are larger than $1/n$. If this fails, there are slightly more refined versions comparing the terms to $1/(n\log(n))$ instead. If the terms of the sum do not come in decreasing order, you can always reorder them so that they do. Then the question of convergence becomes one about how many relatively large terms are in your sum.

For sums that are not absolutely convergent, but may be conditionally convergent, their convergence will depend on cancellation between positive an negative terms in the sum. Perhaps there is a way for example to pair off adjacent positive and negative terms so that the resulting sum is absolutely convergent.

1995 A2: For what pairs (a, b) of positive real numbers does the improper integral

$$\int_b^\infty \left(\sqrt{\sqrt{x+a}-\sqrt{x}}-\sqrt{\sqrt{x}-\sqrt{x-b}}\right)dx$$

converge?

3 Oscillatory Integrals

One kind of improper integral that often converges is one that changes sign faster and faster. In particular if the integral contains a term of the form $\sin(f(x)), \cos(f(x))$ or $e^{if(x)}$ with f growing faster that linearly, the integral will very often converge. One way to show this is integration by parts. For example,

$$\int_a^b e^{if(x)}dx = \int_a^b \frac{if'(x)}{if'(x)}e^{if(x)}dx = \frac{e^{if(x)}}{if(x)}\Big|_a^b - \int_a^b \frac{e^{if(x)}if''(x)}{f'(x)^2}dx.$$

As $f'(x)$ is (hopefully) large (perhaps even relative to $f''(x)$), these new terms are presumably small. Iterating this technique as necessary will often allow one to prove convergence.

2000 A4: Show that the improper integral

$$\lim_{B\to\infty}\int_0^B \sin(x)\sin(x^2)dx$$

converges.

4 The Mean Value Theorem and Rolle's Theorem

The *Mean Value Theorem* says that if f is an everywhere differentiable function, then somewhere between two points a and b, must be a point c where $f'(c)$ is equal to the average slope between these two points of $(f(b) - f(a))/(b - a)$. If $f(a) = f(b) = 0$, we have Rolle's Theorem, which says that between any two zeroes of such a function f must lie a zero of f'. This is often a useful way to prove the existence of such zeroes. Furthermore, if f has $k + 1$ zeroes and is k times differentiable, we can iterate this to find a zero of the kth derivative of f.

2015 B1: Let f be a three times differentiable function (defined on $\mathbb{R}$ and real-valued) such that f has at least five distinct real zeros. Prove that $f + 6f' + 12f'' + 8f'''$ has at least two distinct real zeros.

5 Taking Logs of Products

When trying to analyze infinite products, it is often convenient to take the logarithm of the terms. This relates convergence of the product to convergence of a relevant sum. In particular, if you have $x_i \geq 1$ then thet infinite product

$$\prod_{i=1}^{\infty} x_i$$

converges if and only if the sum

$$\sum_{i=1}^{\infty}(x_i - 1)$$

does.

1970 B1: Evaluate

$$\lim_{n\to\infty} \frac{1}{n^4} \prod_{i=1}^{2n}(n^2 + i^2)^{1/n}.$$