# Math 96:
# Combinatorics Techniques

November 17th, 2023

## 1 Introduction

Combinatorics is the study of discrete with an emphasis on things like counting. It covers a broad range of topics, and we will discuss a few here.

## 2 Basic Counting Principles

A lot of basic counting problems can be solved by breaking down the problem in the correct way.

Firstly, the addition rule says that if $A$ and $B$ are disjoint sets, then $|A \cup B| = |A| + |B|$. This says that if you can split the set of things you are trying to count into mutually exclusive cases, then you can count each case separately and add up the results. More generally, if $A$ and $B$ might overlap, you have $|A \cup B| = |A| + |B| - |A \cap B|$. For more than two sets, this can be generalized using the *Inclusion-Exclusion Principle.*

Secondly, the multiplication rule says that for sets $A$ and $B$ that $|A \times B| = |A||B|$. In other words, the number of ways to pick a pair of an element from $A$ and an element from $B$ is the product of their sizes. More generally, if you can define an object by making a pair of choices the number of ways to do this is equal to the product of the number of ways to make the first choice times the number of ways to make the second choice after having made the first (though this only applies if the number of possibilities for the second choice does not depend on which particular option you have picked for your first choice).

**1983 A1:** How many positive integers $n$ are there such that $n$ is an exact divisor of at least one of the numbers $10^{40}, 20^{30}$?

## 3 Pigeonhole Principle

The *Pigeonhole Principle* states that if $n$ things are sorted into $m$ categories and $n > m$ then at least one category will have more than one thing in it. More generally if $n > km$ for some integer $k$ than some category will have at least $k+1$ things in it. This turns out to be surprisingly useful.

It can be used to show that if you have enough objects some pair of them must be similar. Just define your categories so that any two objects in the same category must be similar in the relevant sense, and be careful about the total number of objects and categories used.

**1978 A6:** Let $n$ distinct points in the plane be given. Prove that fewer than $2n^{3/2}$ pairs of them are unit distance apart.

# 4 Binomial Coefficients

The *binomial coefficient* $\binom{n}{k}$ (pronounced '$n$ choose $k$') is the number of ways to select a $k$ element subset from a set of size $n$. They can be computed using the formula:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

They also satisfy several useful relations:

$$\binom{n}{k} = \binom{n}{n-k}, \quad \binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

Also, the *Binomial Theorem*, which says that

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

**1983 A4:** Let $k$ be a positive integer and let $m = 6k-1$. Let

$$S(m) = \sum_{j=1}^{2k-1} (-1)^{j+1} \binom{m}{3j-1}.$$

For example with $k=3$,

$$S(17) = \binom{17}{2} - \binom{17}{5} + \binom{17}{8} - \binom{17}{11} + \binom{17}{14}.$$

Prove that $S(m)$ is never zero.

# 5 Generating Functions

As was seen in the problem above, you can often get a lot of mileage by way of what are known as generating functions. The idea is that instead of studying a sequence $a_0, a_1, a_2, \ldots$, one can instead consider the power series $A(x) = a_0 + a_1 x + a_2 x^2 + \ldots$. There is essentially a $1-1$ correspondence between sequences and power series and considering the generating function often turns complicated combinatorial questions about the sequence into straightforward algebraic questions about the power series.

A particularly useful thing to keep in mind here is the formula for a product of power series. If $A(x) = \sum_n a_n x^n$, $B(x) = \sum_m b_m x^m$ and $C(x) = A(x)B(x) = \sum_k c_k x^k$, then we have that

$$c_k = \sum_{n+m=k} a_n b_m.$$

The coefficients of the product often count the number of ways that numbers can add up to $k$. In particular if $A(x) = \sum_{n \in S} x^n$ and $B(x) = \sum_{m \in T} x^m$ for some sets $S$ and $T$ of integers, then $c_k$ is the number of ways to write $k$ as the sum of an element of $S$ plus an element of $T$.

**2003 A6:** For a set $S$ of nonnegative integers, let $r_S(n)$ denote the number of ordered pairs $(s_1, s_2)$ such that $s_1 \in S, s_2 \in S, s_1 \neq s_2$ and $s_1 + s_2 = n$. Is it possible to partition the nonnegative integers into two sets $A$ and $B$ in such a way that $r_A(n) = r_B(n)$ for all $n$?