Math 96:
Combinatorics Techniques

November 4th, 2022

Combinatorics is a very broad area, covering a lot of discrete mathematics. As such, it is hard to give a productive overview of all the useful tricks, but here is a quick discussion of some of them.

1 Function Iteration

This is not really a combinatorial topic, but I shoved it here anyway. Function iteration arises when you have a function f mapping a set X to itself. Given an element $x \in X$ one considers the sequence of values $x, f(x), f(f(x)), f(f(f(x))), \ldots$. In particular, we can define $f^{(k)}(x)$ as the value obtained by applying f a total of k times to the point x. The behavior of this sequence can be quite complicated depending on the choice of the function f, but most problems about function iteration involve instances that are somewhat simpler. Here are a few cases that are relatively easy to understand:

1.1 Finite X - Eventually Periodic

If $|X|$ is finite, applying the pigeonhole principle to $x, f(x), f(f(x)), \ldots$ we find that some pair of values must eventually be equal. So eventually, we have that $f^{(i)}(x) = f^{(j)}(x)$ for some $j > i$. Letting $y = f^{(i)}(x)$ and $k = j - i$, we see that $f^{(k)}(y) = y$, and so the sequence of values repeats every k terms. In general, you might see several different values before you eventually hit this repeating loop, but in the special case where f is injective (i.e. it is a permutation of X), then the sequence must eventually loop back to where it started.

1.2 Convergence to a Fixed Point

Another common behavior is where the sequence of values $f^{(k)}(x)$ converges to a value y. If f is continuous, taking the limit of $f(f^{(k)}(x)) = f^{(k+1)}(x)$ yields $f(y) = y$, or that y is a fixed point of f. There are two common circumstances where you can show this happens:

Monotonic Functions: Suppose that f is a strictly increasing function on $\mathbb{R}$ (or a subset thereof), and suppose that $f(x) > x$. Because f is increasing and $f(x) > x$, we have $f(f(x)) > f(x)$ and $f(f(f(x))) > f(f(x))$. By induction we can show that the sequence of values $f^{(k)}(x)$ is increasing in k. This implies that the sequence $f^{(k)}(x)$ must have a limit y, which must by the argument above be a fixed point of f. As for which fixed point, if z is the smallest fixed point of f that is bigger than x, it is not hard to see that if $f^{(k)}(x) < z$ then $f^{(k+1)}(x) < z$ and so by induction $f^{(k)}(x) < z$ for all k. Thus, the limit is at most z, and therefore, z itself is the only possible limit.

Contraction Mapping: Another way to show that $f^{(k)}(x)$ converges to a fixed point y is if f is a contraction mapping. This means that for any x, x' that $\|f(x) - f(x')\| \leq C\|x - x'\|$ for some constant $C < 1$. If this holds then $\|f^{(k+1)}(x) - f^{(k)}(x)\| \leq C^k \|f(x) - x\|$. This allows one to show that the sequence $x, f(x), f(f(x)), \ldots$ is a Cauchy Sequence, and thus (as long as X is complete) has a limit. This will in particular be true if f is a differentiable function and the gradient of f has absolute value at most C everywhere on X. In fact in this case, there must be a *unique* fixed point.

1952 B7: Given any real number N_0, if $N_{j+1} = \cos(N_j)$, prove that $\lim_{j\to\infty} N_j$ exists and is independent of N_0.

2 Basic Counting Techniques

Another common theme in combinatorics is counting the number of ways to do things. These problems can often be solved by breaking down the larger problem into smaller ones.

For example, if the objects being counted can be partitioned into multiple disjoint cases, then the total number will be equal to the sum of the number of objects in each case (if the cases are not disjoint, you will need to use Inclusion-Exclusion which is somewhat more complicated).

Next, if an object can be constructed by making a series of choices, then the number of ways to construct the object will often be the product of the number of available options for each choice, however there are a couple of caveats. For one, this only works if your method of construction provides exactly one way of constructing each object that you want to count. If some objects can be constructed in more than one way, they will be overcounted. Secondly, this requires that the number of options available for each choice you make doesn't depend on which previous choices you have made. The exact options available might vary, but the *number* of options cannot if this technique is to be used.

A final useful tool is that of binomial coefficients. The binomial coefficient n choose k (written $\binom{n}{k}$) is the number of ways to select a subset of size k from a set of size n. It is given by the formula:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Binomial coefficients satisfy several useful relations such as

$$\binom{n}{k} = \binom{n}{n-k},$$

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k},$$

and the Binomial Theorem, which states that

$$(x+y)^n = \sum_{k=0}^{n} x^k y^{n-k} \binom{n}{k}.$$

2007 A3: Let k be a positive integer. Suppose that the integers $1, 2, 3, \ldots, 3k+1$ are written down in random order. What is the probability that at no time during this process, the sum of the integers that have been written up to that time is a positive integer divisible by 3? Your answer should be in closed form, but may include factorials.

3 Basic Generating Functions

Generating functions are a complicated idea whereby instead of studying a sequence of numbers $a_0, a_1, a_2, \ldots$ directly, one instead studies the power series

$$\sum_{n=0}^{\infty} a_n x^n \text{ or } \sum_{n=0}^{\infty} a_n x^n / n!$$

This often allows one to turn combinatorial questions into algebraic ones.

One important fact is that two such power series are the same if and only if their corresponding coefficients are all the same. Some basic generating functions worth knowing include:

$$\frac{1}{1-cx} = \sum_{n=0}^{\infty} c^n x^n,$$

$$\frac{1}{(1-x)^{k+1}} = \sum_{n=0}^{\infty} \binom{n+k}{k} x^n,$$

$$e^{cx} = \sum_{n=0}^{\infty} c^n x^n / n!$$

There are also several useful ways to manipulate these functions. If $A(x) = \sum_{n=0}^{\infty} a_n x^n$ and $B(x) = \sum_{n=0}^{\infty} b_n x^n$, then

$$xA(x) = \sum_{n=1}^{\infty} a_{n-1} x^n,$$

$$A'(x) = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n,$$

$$cA(x) = \sum_{n=0}^{\infty} c a_n x^n,$$

$$A(x) + B(x) = \sum_{n=0}^{\infty} (a_n + b_n) x^n.$$

For exponential generating functions, if $A(x) = \sum_{n=0}^{\infty} a_n x^n/n!$ and $B(x) = \sum_{n=0}^{\infty} b_n x^n/n!$, then

$$xA(x) = \sum_{n=0}^{\infty} n a_{n-1} x^n/n!,$$

$$A'(x) = \sum_{n=0}^{\infty} a_{n+1} x^n/n!,$$

$$cA(x) = \sum_{n=0}^{\infty} c a_n x^n/n!,$$

$$A(x) + B(x) = \sum_{n=0}^{\infty} (a_n + b_n) x^n/n!.$$

1967 A2: Define S_0 to be 1. For $n \geq 1$, let S_n be the number of $n \times n$ matrices whose elements are non-negative integers with the property that $a_{ij} = a_{ji}$, $(i, j = 1, 2, \ldots, n)$ and where $\sum_{i=1}^{n} a_{i,j} = 1$, $(j = 1, 2, \ldots, n)$. Prove

(a) $S_{n+1} = S_n + nS_{n-1}$,

(b) $\sum_{n=0}^{\infty} S_n \frac{x^n}{n!} = \exp(x + x^2/2)$.

4 Products of Generating Functions

A much more powerful application of generating functions comes when considering their products. In particular suppose that you have a generating function $F(x) = \sum_{a \in A} x^{f(a)}$. That is it is the sum over all elements a in some set A of x raised to the power of $f(a)$ for some function $f : A \to \mathbb{N}$. The coefficients of this function are $a_n = \#\{a \in A : f(a) = n\}$. If you also similarly define $G(x) = \sum_{b \in B} x^{g(b)}$, their product is given by the formula:

$$F(x)G(x) = \sum_{a \in A, b \in B} x^{f(a)+g(b)} = \sum_{n=0}^{\infty} x^n \#\{(a, b) : a \in A, b \in B, f(a)+g(b) = n\}.$$

In other words, the coefficients of the product count the number of ways to find a pair of elements so that $f(a) + g(b) = n$.

1987 A6: For each positive integer n, let $a(n)$ be the number of zeroes in the base 3 representation of n. For which positive real numbers x does the series

$$\sum_{n=1}^{\infty} \frac{x^{a(n)}}{n^3}$$

converge?