Math 96:
Calculus Techniques

October 1st, 2021

Today we will discuss some techniques from calculus that may prove useful on the Putnam Exam.

1 Approximating Sums by Integrals

Integrals are essentially continuous versions of summations. This means that the two often give similar results. In particular, for a function f it is often the case that

$$\sum_{k=1}^{n} f(n) \approx \int_0^n f(x)dx.$$

This result is often very useful because the integral is usually easier to compute than the sum. In order to prove this, assume for example that f is an increasing function. We have that

$$\sum_{k=1}^{n} f(n) = \int_0^n f(\lceil x \rceil)dx.$$

Since f is increasing, it follows that

$$\int_0^n f(x)dx \leq \sum_{k=1}^{n} f(n) \leq \int_0^n f(x+1)dx.$$

Note that the lefthand and righthand sides of the above equation differ by at most $f(n+1)$, which is hopefully not too large relative to the whole sum.

A particular instance of this kind of problem is that of Riemann integrals. If a function f is Riemann integrable on an interval (a, b), then the integral is the limit as $n \to \infty$ of the sum of the areas of rectangles of width $1/n$ fitting under the curve $(x, f(x))$. In particular, this means that

$$\int_a^b f(x)dx = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n(b-a)} f(a + k/n).$$

1961 A3: Evaluate

$$\lim_{n\to\infty} \sum_{j=1}^{n^2} \frac{n}{n^2 + j^2}.$$

2 Taylor Expansions

For well-behaved functions f it is the case that

$$f(x) = f(x_0) + (x - x_0)f'(x_0) + (x - x_0)^2/2f''(x_0) + (x - x_0)^3/6f'''(x_0) + \ldots$$

This is often useful is you want to understand the behavior of $f(x)$ for x close to x_0. Often you want to only use the first few terms of this series, but fortunately, there is a way to do this. In particular, we have that for any n times continuously differentiable function f

$$f(x) = \sum_{k=0}^{n-1} (x - x_0)^k / k! f^{(k)}(x_0) + (x - x_0)^n / n! f^{(n)}(\zeta)$$

for some ζ between x and x_0.

It is often convenient to think of this error as $O((x - x_0)^n)$, using the so-called big-O notation, where $O(X)$ denotes a quantity that is at most some (perhaps large) constant times X.

1961 A3: Let $0 < x_1 < 1$ and $x_{n+1} = x_n(1 - x_n)$ for $n = 1, 2, 3, \ldots$. Show that

$$\lim_{n\to\infty} n x_n = 1.$$

3 Finding the Main Term

When trying to approximate sums and integrals it is often useful to start by finding the largest term being summed or integrated. It is often the case (particularly when the summand/integrand is exponentially large), that you can show that your sum or integral is well approximated by just a few terms nearby this largest term.

2011 A3: Find a real number c and a positive number L so that

$$\lim_{r\to\infty} \frac{r^c \int_0^{2\pi} x^r \sin(x) dx}{\int_0^{2\pi} x^r \cos(x) dx} = L.$$

4 Inequalities

There are a lot of techniques involved in solving inequality problems. One technique is to treat them as an optimization problem and use standard calculus techniques. Another often easier method is to use known inequalities to derive your final result. For this technique, here are a few that are worth knowing:

AM-GM For any $a_1, a_2, \ldots, a_n > 0$,

$$\frac{a_1 + a_2 + \ldots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \cdots a_n}.$$

Cauchy-Schwartz For any real numbers $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$,

$$\left(\sum_{i=1}^n a_i b_i\right)^2 \leq \left(\sum_{i=1}^n a_i^2\right)\left(\sum_{i=1}^n b_i^2\right).$$

Jensen's Inequality For any convex function f (i.e. function where $f''(x) \geq 0$ everywhere) and any real numbers $a_1, a_2, \ldots, a_n$, we have that

$$f\left(\frac{a_1 + a_2 + \ldots + a_n}{n}\right) \leq \frac{f(a_1) + f(a_2) + \ldots + f(a_n)}{n}.$$

1961 B1: Let $\alpha_1, \alpha_2, \alpha_3, \ldots$ be a sequence of positive real numbers; define s_n as $(\alpha_1 + \alpha_2 + \ldots + \alpha_n)/n$ and r_n as $(\alpha_1^{-1} + \alpha_2^{-1} + \ldots + \alpha_n^{-1})/n$. Given that $\lim s_n$ and $\lim r_n$ exist as $n \to \infty$, prove that the product is not less than 1.