Math 96:
Recurrences and Functional Equations

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1 Recurrence Relations

A *recurrence relation* is a way of defining a sequence of numbers by making each term (perhaps after some initial few) a function of previous ones. For example, the Fibonacci Sequence is defined by $F_1 = F_2 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for $n \geq 0$. Recurrence relations and their solutions show up not infrequently in contest problems.

For those of you who have taken differential equations, there is a good analogy between recurrence relations and differential equations. Both solve for functions (of real numbers or of integers). The derivatives in differential equations are somewhat like the next terms in the sequence that you see in recurrence relations. Because of this many techniques from differential equations apply to solving recurrence relations.

1.1 Homogeneous Constant Coefficients

Perhaps the most fundamental kind of recurrence relation are called linear homogenous recurrence relations with constant coefficients. These are of the form:

$$x_{n+k} + a_{k-1}x_{n+k-1} + a_{k-2}x_{n+k-2} + \ldots + a_0x_n = 0. \qquad (1)$$

To solve this we relate it to its characteristic polynomial

$$p(t) = t^k + a_{k-1}t^{k-1} + a_{k-2}t^{k-2} + \ldots + a_0.$$

It is not hard to see that if r is a (complex) root of p (i.e. $p(r) = 0$), then $x_n = r^n$ is a solution to Equation (1). Additionally, if r is an order m root we also have solutions $x_n = nr^n$, or $x_n = n^2r^n$ through $x_n = n^{m-1}r^n$. In fact, you can show that the general solution to Equation (1) is of the form

$$x_n = \sum_{i=1}^{k} a_i r_i^n$$

where the r_i are the roots of p and the a_i are constants. This is not too hard to prove by finding a_i so that the first k terms agree, and using induction to prove the rest.
Example: Prove that $F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$ for all n.
1939 B3: Given the power series

$$a_0 + a_1 x + a_2 x^2 + \dots$$

in which

$$a_n = (n^2+1)3^n,$$

show that there is a relation of the form

$$a_n + p a_{n+1} + q a_{n+2} + r a_{n+3} = 0,$$

in which p, q, r are constants independent of n. Find these constants and the sum of the power series.

1.2 Substitution

Another common technique is substitution. Sometimes replacing x_n by some function $f(x_n)$ makes the recurrence relation easier to figure out.
Example: Solve the recurrence relation $x_{n+1} = 2x_n^2 - 1$.

1.3 First Order Relations

There is a nice method for solving recurrence relations of the following form:

$$x_{n+1} = a_n x_n + b_n$$

for fixed sequences a_n and b_n. Letting $A_n = \prod_{i=1}^n a_i$, we have that

$$x_{n+1}/A_n = x_n/A_{n-1} + b_n/A_n.$$

Letting $y_n = x_n/A_{n-1}$, we find that

$$y_{n+1} = y_n + b_n/A_n = y_{n-1} + b_n/A_n + b_{n-1}/A_{n-1} = \dots$$

This gives

$$x_n = A_{n-1} y_n = A_{n-1}(b_{n-1}/A_{n-1} + b_{n-2}/A_{n-2} + \dots + b_1/A_1 + y_1).$$

Example: Letting D_n be the number of derangements of n. Given that $D_n = nD_{n-1} + (-1)^n$ and $D_1 = 0$, show that

$$D_n = n! - (n-1)! + (n-2)! - \dots \pm 0!$$

2 Functional Equations

In a functional equation you are asked to solve for a function given some relationships between its value. While similar to a recurrence relation, a functional equation will generally depend on more than just a few sequential inputs and may even do complicated things like call the function on itself. Functional equation problems are somewhat of a mixed bag, but there are several general techniques that often help.

2.1 Guess the Answer

One of the first things that you should do with functional equation problems is to try to guess what the answer might be. Although this guess will not constitute a proof knowing what exactly you are trying to prove, will often prove immensely useful.

2.2 Clever Substitutions

A standard technique for functional equations, especially those that involve applying the function to itself is to find the right values of the input variables that cause things to simplify. Often taking $x = 0$ or 1 or $f(x) = 0$ (assuming you can prove that there is some x so that this holds) or $x = -y$ will simplify things substantially.

2.3 Algebra

In some functional equation problems, especially those that do not involve composing the function with itself, you can often think of the whole problem as an algebra problem where the "variables" are the values of the function.
1959 A3: Find all functions f of a complex variable such that $f(z)+zf(1-z) = 1 + z$ for all z.

2.4 Uses of Continuity

Some problems tell you that the function you are working with is continuous. There are several ways to exploit this information. For one, continuous functions satisfy the intermediate value theorem, so if your function takes two different values, it must also take all values in between them. If additionally, you can show that your function is injective, (if it is from $\mathbb{R}$ to $\mathbb{R}$) this implies that your function must be monotone (i.e. either strictly increasing or strictly decreasing). Finally, continuity means that if you can determine the values of the function at the rational numbers (or on any other dense subset of the reals), you can use continuity to determine its value everywhere.
1979 A2: For what real k can we find a continuous function $f : \mathbb{R} \to \mathbb{R}$ so that $f(f(x)) = kx^9$ for all x?

2001 B5: Let a and b be real numbers in the interval $(0, 1/2)$, and let g be a continuous real-valued function such that $g(g(x)) = ag(x) + b$ for all real x. Prove that $g(x) = cx$ for some constant c.