

Math 96: Polynomials Techniques

November 1st, 2019

1 Introduction

A polynomial in one variable is a function $p(x) = a_k x^k + a_{k-1} x^{k-1} + \dots + a_0$ for some numbers a_0, a_1, \dots, a_k . These can be integers, real numbers, or even complex numbers. Polynomials can be added, subtracted and multiplied. They can also be factored, and the factorization of a polynomial often conveys useful information about it.

2 Over the Complex Numbers

The *Fundamental Theorem of Algebra* states that any polynomial $p(x) = a_k x^k + a_{k-1} x^{k-1} + \dots + a_0$ over the complex numbers can always be factored as $a_k(x - r_1)(x - r_2) \cdots (x - r_k)$. The r_i are the *roots* of the polynomial p (the values where $p(r) = 0$) and are unique up to reordering of the factors. This has several important consequences:

- A polynomial of degree- d has at most d roots.
 - If f and g are two degree- d polynomials where $f(x) = g(x)$ for at least $d + 1$ different values of x , then f must equal g everywhere (since $f - g$ would otherwise have too many roots).
 - In fact if you specify $d + 1$ pairs of values (x_i, y_i) , there is a *unique* degree- d polynomial f with $f(x_i) = y_i$ for all i .
- A polynomial f divides another polynomial g if and only if all roots of f are also roots of g (with at least the same multiplicity).
- If $f(x) = x^k + a_{k-1} x^{k-1} + \dots + a_0$ has roots r_1, r_2, \dots, r_k , then $a_{k-i} = (-1)^i \sigma_i$ where σ_i is the *i^{th} -symmetric polynomial* in the r 's, namely the sum of all products of k of them.

1939 A3: Find the cubic equation whose roots are the cubes of the roots of

$$x^3 + ax^2 + bx + c.$$

3 Over the Real Numbers

A polynomial with real coefficients is not guaranteed to have the full number of real roots (just consider $x^2 + 1$). However, the complex roots will always come in conjugate pairs ($a + bi$ and $a - bi$). The two factors coming from these roots will give a single quadratic factor. Therefore, a polynomial with real coefficients can always be factored as a product of:

- Linear terms: $x - r$ for a real root r .
- Quadratic terms: $x^2 + ax + b$ with no real root. The lack of a real root here is equivalent to the discriminant $a^2 - 4b$ being negative.

1999 A2: Let $p(x)$ be a polynomial that is non-negative for all real x . Prove that for some k there are polynomials $f_1(x), f_2(x), \dots, f_k(x)$ such that

$$p(x) = \sum_{j=1}^k (f_j(x))^2.$$

4 Over the Rational Numbers of the Integers

Polynomials with rational coefficients or with integer coefficients no longer have such a clean theory of factorization. However, it is still the case that every polynomial has a factorization into irreducible factors (i.e. into polynomials that cannot be factored further over the rationals), and that this factorization is unique up to reordering of the factors and moving constant multiples around (so for example x^2 could be written as either $x \cdot x$ or $(2x)(x/2)$).

One set of polynomials of particular interest are the cyclotomic polynomials $\Phi_n(x)$. These appear in the factorization of $x^n - 1$. In particular $\Phi_n(x)$ has roots that are exactly the *primitive n^{th}* roots of unity (that is the complex numbers x so that $x^n = 1$ but $x^m \neq 1$ for all $0 < m < n$). The cyclotomic polynomial $\Phi_n(x)$ has degree $\varphi(n)$, which is the number of integers $0 < m \leq n$ so that m is relatively prime to n . Furthermore, $x^n - 1$ is the product of $\Phi_d(x)$ over all $d|n$.

1974 B3: Prove that if α is the real number such that $\cos(\pi\alpha) = 1/3$ then α is irrational.

5 Polynomials as Functions

It is also often useful to think of polynomials as functions. This allows many tools of calculus like the intermediate value theorem and mean value theorem to be applied to them.

2014 B4: Show that for each positive integer n , all roots of the polynomial

$$\sum_{k=0}^n 2^{k(n-k)} x^k$$

are real numbers.