Math 96:
Combinatorics Techniques

November 15th, 2019

Combinatorics tends to be a bit of a grab-bag. Most foundationally, combinatorics deals with problems that involve counting of some kind, but the term tends to also cover a pretty wide range of discrete mathematics topics.

1 The Pigeonhole Principle

The pigeonhole principle states that if you have more than n items placed into n categories, then some category must contain at least two of the items. This simple-sounding result can be surprisingly powerful. **1994 A6:** Let $f_1, \ldots, f_{10}$

be bijections of the integers such that for each integer n, there is some composition $f_{i_1} \circ f_{i_2} \circ \cdots \circ f_{i_m}$ of these functions (allowing repetitions) that maps 0 to n. Consider the set of 1024 functions

$$\mathfrak{F} = \{f_1^{e_1} \circ f_2^{e_2} \circ \cdots \circ f_{10}^{c_1 0}\},$$

$e_i = 0$ or 1 for $1 \leq i \leq 10$. (f_i^0 is the identity function and $f_i^1 = f_i$.) Show that if A is a nonempty finite set of integers, then at most 512 of the functions of $\mathfrak{F}$ map A to itself.

The pigeonhole principle is often also useful in geometric problems where it can be used to prove results about packing. In particular, if you want to show that it is impossible to pack too many points into a small space without some pair of points being too close to each other, one can often split the space in question into small regions and use the pigeonhole principle to show that at least two points must lie in the same region.

1954 A2: Consider any five points P_1, P_2, P_3, P_4, P_5 in the interior of a square S of side length 1. Denote by d_{ij} the distance between points P_i and P_j. Prove that at least one of the distances d_{ij} is less than $\sqrt{2}/2$. Can $\sqrt{2}/2$ be replaced by a smaller number in this statement?

2 Binomial Coefficients

One of the most useful concepts in combinatorics is that of a binomial coefficient. The number $\binom{n}{k}$ (pronounced "n choose k") is the number of ways of picking k elements from a set of size n. It can be computed using the formula

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

These numbers also satisfy several useful relations. For example,

$$\binom{n}{k} = \binom{n}{n-k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

The last of these allows one to compute the binomial coefficients using Pascal's Triangle. These numbers also appear in the Binomial Theorem, which states that

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

This last theorem is particularly useful when you take $x = y = 1$ or $x = 1$ and $y = -1$. The former tells us that

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

The latter tells us that for $n > 0$

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k = 0.$$

3 Counting Things in Two Different Ways and Linearity of Expectation

One of the most useful tools in combinatorics is counting the same object in two different ways. This gives two formulas for the same number, and thus can be used to prove a number of useful equations. In fact, essentially all of the facts about binomial coefficients above can be proved this way.

Another version of this is what is known as *linearity of expectation*. This says that if you have some quantity given as a $X = A + B$ then the average of X is the same as the average of A plus the average of B. So for example, if a students grade in a class in based on their total score over several tests, the average total score in the class is the sum of the average scores on each of the tests. This is a very versatile technique, especially when you write things in terms of indicator functions (that is a quantity that is 1 if some event happens and 0 if it does not).

1974 A4: An unbiased coin is tossed n times. What is the expected value of $|H - T|$, where H is the number of heads and T is the number of tails? In other words, evaluate *in closed form*

$$\frac{1}{2^{n-1}} \sum_{k<n/2} (n-2k)\binom{n}{k}.$$

4 The Inclusion-Exclusion Principle

Often when trying to count something, the things we are trying to count can be split up into several different cases. If these cases are disjoint, we can simply add up the number of things from each case. However, if the cases are not disjoint, this may lead to overcounting. For example, if we wanted to count the number of students in class with either black hair or blond hair, we could add up the number of students with black hair and the number of students with blond hair. However, if we wanted to count the number of students with black hair or green eyes, then if we simply added up the number of students with black hair and the number with green eyes, we would have counted any students with both black hair and green eyes twice. We can deal with this problem by figuring out the number of students we are overcounting and correcting for it. In this case, if we then subtracted off the number of students with black hair and green eyes, we would be good.

More abstractly, we have two sets A and B and want to count the size of $A \cup B$. If A and B are disjoint, we can take the sum $|A \cup B| = |A| + |B|$. However, if they are not disjoint, we need a correction term $|A \cup B| = |A| + |B| - |A \cap B|$. More generally, if we have n sets $A_1, A_2, \ldots, A_n$ and want to count the number of elements in any of them (i.e. the size of their union), we can do this using what is known as Inclusion-Exclusion. We compute it by first adding up the sizes of the individual sets. We then must subtract off the sizes of the intersections of each pair. Then we need to add back in the sizes of intersections of triples, and subtract off quadruples and so forth. In general, we get the formula,

$$|A_1 \cup A_2 \cup \ldots \cup A_n| = \sum_{k=1}^{n} (-1)^{k+1} \sum_{1 \le i_1 < i_2 < \ldots < i_k \le n} |A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_k}|.$$

2009 B3: Call a set S of $\{1, 2, \ldots, n\}$ *mediocre* if it has the following property: Whenever a and b are elements of S whose average is an integer, that average is also in S. Let $A(n)$ be the number of mediocre subsets of $\{1, 2, \ldots, n\}$. [For instance every subset of $\{1, 2, 3\}$ is mediocre except for $\{1, 3\}$, so $A(3) = 7$.] Find all positive integers n so that $A(n+2) - 2A(n+1) + A(n) = 1$.