# Basics of Proofs

The Putnam is a proof based exam and will expect you to write proofs in your solutions. Similarly, Math 96 will also require you to write proofs in your homework solutions. If you've seen proofs before, you already know what you're getting into and should feel free to ignore this handout. On the other hand, if you are new to writing proofs, this handout is designed to let you know what will be expected of you and to give you some tips on getting started.

## 1 The Basics

A mathematical proof is a convincing argument that some claim is true. Well it's slightly more than that. A proof is a *super convincing* argument that your claim is true. If done correctly, a proof should leave no doubt in a reader's mind (so long as this reader is familiar with the subject matter) of your claim.

Alright. How do we make an argument that convincing? We do this by breaking down our argument into a bunch of simple steps each of which can be properly justified. These justifications can take a range of forms. They can be axioms, definitions, things proved in class, or other things that you have already established in your proof.

So you've found an argument that works, and broken it down into steps. How do you actually write your proof? Think of this part as writing a short essay explaining your argument. Lay out your claims in a clear an organized fashion. Write in complete, grammatically correct English sentences, using equations where necessary. Make sure to define any new terms or variables that you are using. As a good essay will have an introduction, a middle and a conclusion, so should a good proof. Begin by defining your variables and laying out your initial hypotheses (though some of this can often be skipped if it is clear from context), then lay out the meat of your argument, and conclude by reminding the reader what you have proved.

One last thing to keep in mind when writing a proof is the level of detail in which you explain your argument. In practice this sort of thing varies widely depending on what you are trying to prove and who you are writing for. While an expert might consider some steps justified by saying "by the Spectral Theorem" or even "it is easily verified that" a novice may require a more detailed explanation, and proof intended to be computer-verifiable will require a vastly more detailed justification. Thus when writing proofs, we must always try to strike a balance between being too vague by not giving enough details, and being too tedious by giving too many. When writing proofs as part of your coursework, you should attempt to give enough details to convince the grader that you understand what is going on and that you could fill in the rest if asked. When in doubt though, it is always better to add more details.

## 2 Proof Strategies

Finding the right argument to prove a given claim can be a tricky thing. It can require deep understanding, hard work, and a good deal of cleverness. As such, there's no sure way to go about it, and getting good at it depends on a lot of practice and trial and error. So while there's no general technique for proving anything that you need to (and if there were, it would put a lot of mathematicians out of business), here are some tips to get you started (though we will be discussing many more during the course of the class).

### 2.1 Understand What's Going On

A good first step on any problem is to try to get a feel for what's going on. Reread the problem statement. Make sure you know what hypothesis you are being given, and what you are being asked to show with them. Try looking at some simple examples and look for patterns. If you can get some intuition about what kinds of things are going to be true for this problem, it can often be very useful in directing you towards fruitful lines of attack. Finally, if you can get some feel for *why* the claim you are looking at it true, even if it's vague and hard to put into words, this is often a great way to guide your further efforts.

## 2.2 Look for Equivalent Statements

Often problems are not handed to you in their simplest form, and a quick search for other ways of saying the same thing can give you a new problem that is easier to wrap your head around. Are you being asked to prove that a vector space is finite dimensional? Maybe you know a theorem which gives you some nice criteria for when that happens. Are you being asked to prove an implication? Perhaps proving the contrapositive will be easier.

## 2.3 Unrolling Definitions

It's important to remember the basic definitions and what they mean. If a question asks you to show that a given set is a vector subspace, you will need to recall what that means and show that all of the necessary conditions hold. If the problem statement tells you that a given space is a vector space, you are likely going to need to use some of the properties that this implies later on.

## 2.4 Use your Hypothesis

Are you working on a problem and having trouble making progress beyond some point? Do you need some extra thing to be true but don't know how to show it? It might be a good idea to go back and reread the problem statement. Maybe that little bit that you were looking for was actually one of the assumptions that was given to you. On the flip side, if you see some assumptions in the problem statement that seem strange to you, it might be a good idea to think about how they might end up being used. Homework problems tend to be pretty clean and tend not to give you a lot of unnecessary assumptions to work with.

## 2.5 Work on Special Cases

Is the problem proving too difficult for you? Is there some simpler version of it that you can solve? Maybe if you assume that things are finite dimensional or the matrix you are working with is diagonal, things become easier to solve. Although solving the easier version will not immediately solve to original problem, it will often provide you with useful insights into the full solution.

## 2.6 Check your Work

Once you've finished your proof, it's probably a good idea to read it over to make sure it's correct. Remember, a proof is supposed to be a super convincing argument, so if while reading it you find yourself unsure of one of your claims, it's probably a good idea to make sure that it's correct. A good way to do this is often to try filling in some more details of the proof in order to make things more clear.

## 2.7 Specific Techniques

In addition to the above strategies, there are a few specific types of proof that are often useful to know about.

### 2.7.1 Case Analysis

Sometimes it is difficult to come up with a single argument to cover the general case of your problem. You might find one argument that works for invertible linear transformations, but need to find a totally different argument for non-invertible ones. On the other hand, if you have separate arguments that work in each of the above cases, you can combine them to get something that works in general.

Generally speaking, if you can break your problem up into cases and solve each of them individually, you can combine them to get a full solution. When using this technique it is usually a good idea to inform your reader ahead of time that this is what you are doing and which cases you are considering (and at this point you should make sure that the cases you are considering cover all possibilities) and then list the argument for each case separately, clearly laying out which case you are working on where. Here's an example of this technique in action:

**Problem.** *Show that for any real number $x$ that there is some real number $y$ so that $x^2y = x$.*

Solving the above equation for $y$ we find that taking $y = x^{-1}$ should suffice. And this works unless $x$ is 0, in which case it has no inverse. On the other hand, if $x$ is 0, any $y$ should suffice. Thus, in order to prove this we need to consider separately the cases where $x = 0$ and where $x \neq 0$. A good proof might look like this:

*Proof.* We break this problem into cases based on whether or not $x$ equals 0.
**Case 1:** $x \neq 0$
If $x \neq 0$, we may take $y = x^{-1}$. For this choice of $y$, $x^2y = x^2x^{-1} = x$, as desired.
**Case 2:** $x = 0$
If $x = 0$, we may take $y = 0$. For this choice of $y$, $x^2y = 0^20 = 0 = x$, as desired.
Thus, in either case, there is always some $y$ so that $x^2y = x$. □

Let's look at one more example:

**Problem.** *Let $x$ and $y$ be real numbers. Show that*

$$\max(x, y) \geq \frac{x + y}{2}.$$

This just says that the biggest of two numbers is always at least as big as the average. This is clearly true, but it can be tricky to get a handle on this problem. Part of the issue is that the $\max(x, y)$ is hard to deal with directly. On the other hand, it is either $x$ or $y$, so we just need to consider these cases separately.

*Proof.* We consider separately the cases based on which of $x$ and $y$ is larger.
**Case 1:** $x \geq y$
If $x \geq y$ then

$$\max(x, y) = x = \frac{x + x}{2} \geq \frac{x + y}{2}.$$

**Case 2:** $y \geq x$
If $y \geq x$ then

$$\max(x, y) = y = \frac{y + y}{2} \geq \frac{x + y}{2}.$$

Thus in either case $\max(x, y) \geq (x + y)/2$. □

If you look at the two cases in the last proof, you'll note that they are essentially the same as each other. This is because $x$ and $y$ fill identical roles in the problem statement. Thus, the case where $x \geq y$ will look essentially identical to the case where $y \geq x$. By taking note of this fact, we can simplify the above proof by only dealing with one of these two cases and noting that the other one is identical. The way such arguments are made in practice is to assume Without Loss Of Generality (abbreviated WLOG) that $x \geq y$. Note that this only works here because the other case where $y \geq x$ is identical. Using this technique the proof looks like this:

*Proof.* WLOG we assume that $x \geq y$. Thus,

$$\max(x, y) = x = \frac{x + x}{2} \geq \frac{x + y}{2}$$

as desired. □

### 2.7.2 Proof by Contradiction

Sometimes the best way to prove a statement is true is to show that it cannot be false. To do this you assume that your intended statement is false and show that under this assumption you reach a contradiction. This technique is often useful for showing that things with certain properties do not exist, since assuming the negation gives us another useful hypothesis. When writing a proof by contradiction, it is important to tell the reader what you are doing. A standard way of doing this is to start with the line "Assume for sake of contradiction ⟨negation of the statement you are trying to prove⟩", then give your argument leading to a contradiction, point out the contradiction, and then restate your original (and now proven) claim. As an example of how this works, consider the following:

**Problem.** *Show that* $\sqrt{2}$ *is irrational.*

In other words, we need to show that $\sqrt{2}$ cannot be written as $\frac{a}{b}$ for integers $a, b$. This is somewhat difficult to do directly, because we would need to show that such $a, b$ do not exist. It will be much easier to assume that they do and reach a contradiction.

*Proof.* Assume for sake of contradiction that $\sqrt{2}$ is rational. This means that we can write $\sqrt{2}$ as $\frac{a}{b}$, which is some fraction given in lowest terms (this part will be important later). In other words, $\sqrt{2} = \frac{a}{b}$ for some relatively prime integers $a, b$. Multiplying both sides by $b$ and squaring, we find that $a^2 = 2b^2$. Since $a^2$ is even, $a$ must be even, so we may write $a = 2c$ for some other integer $c$. Thus, $2c^2 = b^2$. Since $b^2$ is even $b$ must also be even. On the other hand, since $a$ and $b$ are both even, they are not relatively prime, which contradicts our assumption that $\frac{a}{b}$ was given in lowest terms. Therefore, $\sqrt{2}$ must be irrational, which completes our proof. □

### 2.7.3 Induction

Another useful technique is mathematical induction. The idea here is the following. Suppose that you have some sequence of statements $P(0), P(1), P(2), \ldots$ indexed by integers. Suppose you can show:

1. One of these statements, say $P(m)$ is true (this is known as the base case)
2. For all $n$, if $P(n)$ holds, then $P(n+1)$ is true (this is known as the inductive step)

Then you can conclude that $P(n)$ holds for all $n \geq m$. This is because by 1 we know that $P(m)$ holds. Then by using 2 repeatedly, we can show $P(m+1)$ and then $P(m+2)$ and then $P(m+3)$ and so on are all true. Proofs by induction are often useful when there's a clear relationship between $P(n)$ and $P(n+1)$.

As proof by induction is a complicated procedure, it is important to inform your reader about what's going on. Fortunately, there is a nice standardized way to do this. The outline is as follows:

- Tell the reader what you are proving by induction and which variable you are inducting on.
- State what $m$ is being used as a base case, and prove it.
- Assume $P(n)$ (known as the inductive hypothesis) and use it to prove $P(n+1)$. Note that this completes the inductive step.
- Restate the result that you've just proved.

OK. Let's see this in action:

**Problem.** *Show that for any* $n \geq 1$

$$1 + 2 + \ldots + n = \frac{n(n+1)}{2}.$$

Note that increasing $n$ by 1 changes the above equation only very slightly. This makes it an excellent candidate to prove by induction.

*Proof.* We prove by induction on $n$ that

$$1 + 2 + \ldots + n = \frac{n(n+1)}{2}.$$

**Base Case:** $n = 1$.
If $n = 1$ the above equation reduces to $1 = \frac{1 \cdot 2}{2}$, which is true.
**Inductive Step:** Assume that $1 + 2 + \ldots + n = n(n+1)/2$.
We have that

$$1+2+\ldots+(n+1) = (1+2+\ldots+n)+(n+1) = \frac{n(n+1)}{2}+(n+1) = \frac{(n+1)(n+2)}{2} = \frac{(n+1)((n+1)+1)}{2}.$$

Note that the second equation above follows from out inductive hypothesis. This completes our inductive step.

Thus, we have proved for all $n \geq 1$ that

$$1 + 2 + \ldots + n = \frac{n(n+1)}{2}.$$

As desired. □