Math 96: Polynomials Techniques

October 21st, 2022

A (one-variable) *polynomial* is a function of the form:

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_0$$

for some non-negative integer n and some numbers a_i with $a_n \neq 0$. The number n here is called the *degree* of the polynomial.

More generally, a polynomial in several variables is a function

$$f(x_1, x_2, \ldots, x_k) = \sum_{\alpha} a_\alpha \prod_{i=1}^{k} x_i^{\alpha_i},$$

where the above sum is over finitely many α's whose coefficients are all non-negative integers.

1 Factorization

A polynomial whose coefficients lie in a ring R is called *irreducible* if it cannot be written as a product of lower degree polynomials. Much like integers have unique factorization into primes, polynomials (often) have unique factorization into irreducibles.

Theorem 1. *Let $R = \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ or any field and let f be a polynomial (in any number of variables) with coefficients in R. Then f can be written as a product of irreducible polynomials with coefficients in R and this representation is unique up to reordering of the factors and multiplying/dividing factors by elements of R.*

This theorem is particularly interesting for univariate polynomials when R is $\mathbb{C}$ (or any other algebraically closed field). Then the Fundamental Theorem of Algebra says that f will factor into *linear* factors. In particular, we can write

$$f(x) = a_n \prod_{i=1}^{n} (x - r_i).$$

The r_i's here are the *roots* of f. That is the values of x so that $f(x) = 0$.

Another interesting situation is when you consider a univariate polynomial with real coefficients. Then f will factor entirely into linear and quadratic factors where the quadratic factors have complex roots.

2007 B4: Let n be a positive integer. Find the number of pairs P, Q of polynomials with real coefficients such that

$$(P(X))^2 + (Q(X))^2 = X^{2n} + 1$$

and $\deg P > \deg Q$.

2 Relationship Between Roots and Coefficients

Suppose that you have a polynomial f that factors completely

$$f(x) = (x - r_1)(x - r_2) \cdots (x - r_n).$$

What is the relationship between the roots r_i of f and its coefficients? It turns out that:

$$f(x) = \sum_{k=0}^{n} (-1)^k x^{n-k} \sigma_k$$

where σ_k is the k^{th} *elementary symmetric polynomial* in the r_i. In particular,

$$\sigma_k = \sum_{1 \le a_1 < a_2 < \ldots < a_k \le n} \prod_{i=1}^{k} r_{a_i}$$

is the sum of all products of k of the r_i's. Particularly notable examples of these are

$$\sigma_1 = r_1 + r_2 + \ldots + r_n$$

and

$$\sigma_n = r_1 r_2 \cdots r_n.$$

These polynomials have a number of other uses. For instance, the Fundamental Theorem of Symmetric Polynomials states that any symmetric polynomial in $r_1, \ldots, r_n$ (that is a polynomial in the r_i's that doesn't change when you swap r_i with r_j) can be written as a polynomial in the σ_j's.

1977 A1: Consider all lines which meet the graph of

$$y = 2x^4 + 7x^3 + 3x - 5$$

in four distinct points $(x_i, y_i), i = 1, 2, 3, 4$. Show that

$$\frac{x_1 + x_2 + x_3 + x_4}{4}$$

is independent of the line and find its value.

Note that a similar technique to this can be used given a polynomial and a line and all but one of their intersections, one can often solve for the x-coordinate of the reaming intersection.

3 Interpolation

Note that a (non-zero) degree n polynomial in one variable over a field (or more generally a domain) can have at most n linear factors and therefore can have at most n roots. In particular, this means that any polynomial with *more* than n roots must be identically zero. In particular, this means that if you have two degree-n polynomials f and g so that $f(x) = g(x)$ for at least $n+1$ different values of x, then f and g must be the same. This is often useful if one wants to check equality of two polynomials. Although these polynomials might be quite complicated, if you can find $n+1$ points at which they are easy to evaluate, that is enough to check equality everywhere.

Another way of looking at this is that a degree n polynomial f is *uniquely determined* by its values at any $n+1$ points. In particular if $f(x_i) = y_i$ for all $1 \le i \le n+1$, we have that

$$f(x) = \sum_{i=1}^{n+1} y_i \prod_{1 \le j \le n+1, j \ne i} \frac{x - x_j}{x_i - x_j}.$$

1992 B5: Let D_n denote the value of the $(n-1) \times (n-1)$ determinant

$$\begin{bmatrix} 3 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 4 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 5 & 1 & \cdots & 1 \\ 1 & 1 & 1 & 6 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & 1 & \cdots & n+1 \end{bmatrix}.$$

Is the set $\left\{\frac{D_n}{n!}\right\}_{n \ge 2}$ bounded?

4 Solving for Coefficients

Another useful technique for dealing with polynomials is simply trying to solve for the coefficients of an unknown polynomial, especially when you can do so using systems of linear equations.

1987 A4: Let P be a polynomial, with real coefficients, in three variables and F be a function of two variables such that

$$P(ux, uy, uz) = u^2 F(y - x, z - x) \text{ for all real } x, y, z, u,$$

and such that $P(1,0,0) = 4, P(0,1,0) = 5$, and $P(0,0,1) = 6$. Also let A, B, C be complex numbers with $P(A,B,C) = 0$ and $|B - A| = 10$. Find $|C - A|$.