Number Theory Techniques

Math 96
October 7th
Announcements

• Homework 2 due by email (dakane@ucsd.edu)
• Homework 3 on course webpage. Due next week.
• If you have not already registered to take the Putnam please do so at: https://www.maa.org/math-competitions/putnam-competition
• If you have not completed either of the first two homeworks, please talk to me after class to ensure a certification of commencement of academic activity.
Number Theory

Number theory is the study of the natural numbers: 1, 2, 3, ... and particular of their arithmetic.

• How do natural numbers add or multiply to get other natural numbers?
• Which polynomials have natural number solutions?
Primes

**Definition:** A positive integer more than 1 is **prime** if it cannot be written as the product of two smaller positive integers.
Primes

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So 101 is prime, but 102 = 2\times51 is not.
Theorem: Any positive integer \( n \) can be written as a product of primes. Furthermore, this product is unique up to reordering of the terms.
Fundamental Theorem of Arithmetic

**Theorem:** Any positive integer $n$ can be written as a product of primes. Furthermore, this product is unique up to reordering of the terms.

For example, $60 = 2 \times 2 \times 3 \times 5 = 5 \times 2 \times 3 \times 2$. But it cannot be written as a product of primes except as two $2$s, a $3$ and a $5$. 
Consequence

This means that multiplicatively, a positive integer can be thought of as a bag of primes to be multiplied together. Two integers are the same if and only if they correspond to the same bag of primes.
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This is useful if you want to think about numbers only considering their multiplication.
Other Basic Concepts

• We say that $m$ divides $n$ (or $n$ is a multiple of $m$, denoted $m|n$) if $n = axm$ for some integer $a$. This happens if and only if every prime dividing $m$ also divides $n$ at least as many times.
Other Basic Concepts

• We say that \( m \) divides \( n \) (or \( n \) is a multiple of \( m \), denoted \( m \mid n \)) if \( n = axm \) for some integer \( a \). This happens if and only if every prime dividing \( m \) also divides \( n \) at least as many times.

• The greatest common divisor of \( m \) and \( n \) (\( \gcd(n,m) \)) is the largest integer that divides both. You can get it by taking the product of all the primes that divide both.
Euclidean Algorithm

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Euclidean Algorithm

A more efficient way to compute the gcd of two numbers is by what is known as the Euclidean Algorithm.

It is not hard to see that

\[ \text{gcd}(n,m) = \text{gcd}(n-m,m) = \text{gcd}(n,m-n). \]

You can repeatedly subtract the smaller number from the larger one until they are the same.
Example

gcd(255, 374) =
Example

gcd(255, 374) =

gcd(255, 119) =
Example

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gcd(255, 119) =
gcd(136, 119) =
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gcd(255, 374) =
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17.
**2002 A5:** Define a sequence by \( a_0 = 1 \), together with the rules that \( a_{2n+1} = a_n \) and \( a_{2n+2} = a_n + a_{n+1} \) for each integer \( n \geq 0 \). Prove that every positive rational number appears in the set \( \{ a_{n-1} / a_n : n \geq 1 \} = \{1/1, 1/2, 2/1, 1/3, 3/2, \ldots \} \).
Observation 1

If any positive rational number $q$ is put into reduced form for $q = x/y$, where $x$ and $y$ have no common factors. In particular, $\gcd(x, y) = 1$ (this is known as $x$ and $y$ are relatively prime).
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If any positive rational number $q$ is put into reduced form for $q = \frac{x}{y}$, where $x$ and $y$ have no common factors. In particular, $\gcd(x, y) = 1$ (this is known as $x$ and $y$ are relatively prime).

We need to show that for any relatively prime $x$ and $y$ that there is some $n$ so that $a_{n-1} = x$, and $a_n = y$. 
Observation 2

Suppose that \((a_{n-1}, a_n) = (x, y)\) then:

- \((a_{2n-1}, a_{2n}) = (x, x+y)\)
- \((a_{2n}, a_{2n+1}) = (x+y, y)\)
Observation 2

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- \((a_{2n}, a_{2n+1}) = (x+y, y)\)

These are the reverses of the steps used in the Euclidean algorithm!
Proof sketch

Given $x$ and $y$ relatively prime, we can use the Euclidean algorithm so that starting with $(x,y)$ we apply steps of the form:

- $(x,y) \rightarrow (x, y-x)$
- $(x,y) \rightarrow (x-y, y)$

until we reach $(1,1) = (a_0, a_1)$. 
Proof sketch

Given \( x \) and \( y \) relatively prime, we can use the Euclidean algorithm so that starting with \( (x,y) \) we apply steps of the form:

- \((x,y) \rightarrow (x,y-x)\)
- \((x,y) \rightarrow (x-y,y)\)

until we reach \((1,1) = (a_0,a_1)\).

Applying these steps in the opposite order, we can find \( n \) so that \((a_{n-1},a_n) = (x,y)\).
Example

\((x,y) = (2,5)\):

\((2,5) \rightarrow (2,3) \rightarrow (2,1) \rightarrow (1,1) = (a_0, a_1)\).
Example

\[(x,y) = (2,5):
\]

\[(2,5) \rightarrow (2,3) \rightarrow (2,1) \rightarrow (1,1) = (a_0,a_1).\]

\[(1,1) = (a_0,a_1)\]

\[(2,1) = (a_2,a_3)\]

\[(2,3) = (a_5,a_6)\]

\[(2,5) = (a_{11},a_{12})\]
Modular Arithmetic

Have you noticed that you can determine the last digit of a sum or product of two numbers just by adding or multiplying the two last digits? This gives rise to a kind of “last digit arithmetic” which can be useful. Modular arithmetic is a generalization of this to bases other than 10.
Congruence

**Definition:** We say that two integers $a$ and $b$ are congruent modulo $m$ for some other integer $m$ (denoted $a \equiv b \pmod{m}$) if $a-b$ is a multiple of $m$. 
Basic Facts

• ≡ (mod m) is an equivalence relation:
  – a ≡ a (mod m)
  – a ≡ b (mod m) if and only if b ≡ a (mod m)
  – a ≡ b (mod m) and b ≡ c (mod m) ⇒ a ≡ c (mod m)
Basic Facts

• \( \equiv (\text{mod } m) \) is an equivalence relation:
  – \( a \equiv a \) (mod m)
  – \( a \equiv b \) (mod m) if and only if \( b \equiv a \) (mod m)
  – \( a \equiv b \) (mod m) and \( b \equiv c \) (mod m) \( \Rightarrow \) \( a \equiv c \) (mod m)

• Every integer is congruent modulo m to exactly one of 0, 1, 2, ..., m-1 (by taking the remainder when dividing by m).
More Basic Facts

• If $a \equiv a' \pmod{m}$ and $b \equiv b' \pmod{m}$ then:
  – $a+b \equiv a'+b' \pmod{m}$
  – $ab \equiv a'b' \pmod{m}$
More Basic Facts

• If \( a \equiv a' \pmod{m} \) and \( b \equiv b' \pmod{m} \) then:
  - \( a+b \equiv a'+b' \pmod{m} \)
  - \( ab \equiv a'b' \pmod{m} \)

This essentially says that you can do arithmetic modulo \( m \). That is you can do arithmetic on numbers only ever caring about results modulo \( m \). This gives rise to the ring \( \mathbb{Z}/m \).
Problem

1952 A1: Let

$$f(x) = \sum_{i=0}^{n} a_i x^{n-i}$$

be a polynomial of degree $n$ with integral coefficients. If $a_0$, $a_n$ and $f(1)$ are all odd, prove that $f(x) = 0$ has no rational roots.
Observation 1

Suppose that $f(x) = 0$ has a root at $p/q$. We have that

$$0 = a_0 \frac{p}{q}^n + a_1 \frac{p}{q}^{n-1} + \ldots + a_n.$$
Suppose that $f(x) = 0$ has a root at $p/q$. We have that

$$0 = a_0 (p/q)^n + a_1 (p/q)^{n-1} + \ldots + a_n.$$  

Clearing denominators we have

$$0 = a_0 p^n + a_1 p^{n-1} q + \ldots + a_{n-1} p q^{n-1} + a_n q^n.$$
Observation 2

We can take p and q to be relatively prime, so they are not both even. This means that either:

• p is even and q is odd
• q is even and p is odd
• p and q are both odd
Observation 2

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- \( q \) is even and \( p \) is odd
- \( p \) and \( q \) are both odd

In each case we can consider what happens modulo 2.
p even q odd

\[ a_0 p^n + a_1 p^{n-1} q + \ldots + a_{n-1} p q^{n-1} + a_n q^n \]

is equivalent modulo 2 to

\[ a_0 0^n + a_1 0^{n-1} 1 + \ldots + a_{n-1} 0 1^{n-1} + a_n 1^n \equiv a_n \equiv 1. \]
p even q odd

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Thus, there is no solution of this form.
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is equivalent modulo 2 to

\[ a_0 1^n + a_1 1^{n-1} 1 + \ldots + a_{n-1} 1 \cdot 1^{n-1} + a_n 1^n \equiv a_0 + a_1 + \ldots + a_n \equiv f(1) \equiv 1. \]

Thus, there is no solution of this form.

Therefore, f has no rational roots.
Modular Inverses

You can multiply numbers modulo m, but when can you divide them?
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You can multiply numbers modulo m, but when can you divide them?

In particular, given a modulo m, its inverse, \( a^{-1} \) (mod m) should be some number b so that \( ab \equiv 1 \) (mod m).
Modular Inverses

This is not always possible. For example you cannot divide by 0. There is also no number b so that $2b \equiv 1 \pmod{4}$ because $2b$ will always be even.
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This is not always possible. For example you cannot divide by 0. There is also no number \( b \) so that \( 2b \equiv 1 \pmod{4} \) because \( 2b \) will always be even.

However, if \( a \) and \( m \) are relatively prime, there will always be a unique inverse of \( a \) modulo \( m \). In particular if \( m \) is prime and \( a \not\equiv 0 \pmod{m} \), then \( a \) has an inverse mod \( m \).
1957 B1: Consider the determinant $|a_{ij}|$ of order 100 with $a_{ij} = i \times j$. Prove that if the absolute value of each of the $100!$ terms in the expansion of this determinant is divided by 101 then the remainder in each case is 1.
Observation 1

What is a term in the determinant? We take 100 entries that have one from each row and one from each column and multiply them together.
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What is the resulting product?
Writing each $a_{ij} = i \times j$, we get the product of all of the rows times the product of all of the columns.
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What is a term in the determinant?
We take 100 entries that have one from each row and one from each column and multiply them together.

What is the resulting product?
Writing each $a_{ij} = i \times j$, we get the product of all of the rows times the product of all of the columns.

Each term gives $(1 \cdot 2 \cdot 3 \cdot \ldots \cdot 100)(1 \cdot 2 \cdot 3 \cdot \ldots \cdot 100)$. 
Observation 2

101 is prime.
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Each term in the first part of the product has an inverse in the second part. Rearranging, this product modulo 101 equals:

\[(11^{-1}) (22^{-1}) (33^{-1})... ((100)(100)^{-1}) \equiv 1^{100} \equiv 1.\]
Observation 2

101 is prime.

Each term in the first part of the product has an inverse in the second part. Rearranging, this product modulo 101 equals:

\[(11^{-1}) (22^{-1}) (33^{-1})... ((100)(100)^{-1}) \equiv 1^{100} \equiv 1.\]

**Note:** Using similar ideas you can prove Wilson’s Theorem, which states that for any prime \(p\) that \((p-1)! \equiv -1 \pmod{p} \).
Modular Exponents

What are the powers of 3 modulo 7?

\[
\begin{align*}
3^0 &= 1 & 3^6 &= 729 \equiv 1 \\
3^1 &= 3 & 3^7 &= 2187 \equiv 3 \\
3^2 &= 9 \equiv 2 & 3^8 &= 6561 \equiv 2 \\
3^3 &= 27 \equiv 6 & 3^9 &= 19683 \equiv 6 \\
3^4 &= 81 \equiv 4 & 3^{10} &= 59049 \equiv 4 \\
3^5 &= 243 \equiv 5 & 3^{11} &= 177147 \equiv 5
\end{align*}
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\end{align*}
\]...

...and it repeats. This is not uncommon.
Fermat’s Little Theorem

**Theorem:** Let $p$ be a prime and $a$ be relatively prime to $p$. Then

$$a^{p-1} \equiv 1 \pmod{p}.$$
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**Theorem:** Let $p$ be a prime and $a$ be relatively prime to $p$. Then

$$a^{p-1} \equiv 1 \pmod{p}.$$ 

This also means that $a^k \equiv a^{k+(p-1)} \equiv a^{k+2(p-1)}$. In fact, $a^k$ modulo $p$ will only depend on what $k$ is modulo $p-1$. 
Order of an Element

So the powers of a modulo \( p \) repeat every \( p-1 \) terms, but they may repeat more often than that.
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So the powers of a modulo \( p \) repeat every \( p - 1 \) terms, but they may repeat more often than that.

We define \( \text{ord}_p(a) \) to be the smallest positive integer \( k \) so that \( a^k \equiv 1 \pmod{p} \). It is not hard to see that the powers of a repeat every \( \text{ord}_p(a) \) terms, but only repeat that often.
Order of an Element

So the powers of a modulo $p$ repeat every $p-1$ terms, but they may repeat more often than that.

We define $\text{ord}_p(a)$ to be the smallest positive integer $k$ so that $a^k \equiv 1 \pmod{p}$. It is not hard to see that the powers of $a$ repeat every $\text{ord}_p(a)$ terms, but only repeat that often.

It must also be the case that $\text{ord}_p(a)$ divides $p-1$ since $a^{p-1} \equiv a^0 \pmod{p}$. 
Primitive Roots

Also, interestingly, modulo every prime $p$ it is possible to find a primitive root. That is some number $g$ modulo $p$ so that $\text{ord}_p(g) = p-1$. 
Primitive Roots

Also, interestingly, modulo every prime $p$ it is possible to find a **primitive root**. That is some number $g$ modulo $p$ so that $\text{ord}_p(g) = p-1$. This means that $g, g^2, g^3, \ldots, g^{p-1}$ must be exactly the numbers $1, 2, 3, \ldots, p-1$ modulo $p$. That is every number modulo $p$ (except for 0) can be written as a power of $g$. 
1972 A5: Show that if $n$ is an integer greater than 1, then $n$ does not divide $2^n - 1$. 
Observation 1

Assume for sake of contradiction that $n | 2^n - 1$.

Let $p$ be a prime dividing $n$. 
Observation 1

Assume for sake of contradiction that \( n \mid 2^n - 1 \).

Let \( p \) be a prime dividing \( n \).

- Since \( p \) divides \( n \) and \( n \) divides \( 2^n - 1 \), \( p \) must divide \( 2^n - 1 \).
  - Note: \( p \) cannot be 2.
Observation 1

Assume for sake of contradiction that $n \mid 2^n - 1$.

Let $p$ be a prime dividing $n$.

• Since $p$ divides $n$ and $n$ divides $2^n - 1$, $p$ must divide $2^n - 1$.
  – Note: $p$ cannot be 2.

• $2^n \equiv 1 \pmod{p}$
  – This means that $n$ is a multiple of $\text{ord}_p(2)$. 
Observation 2

What if \( p \) is the smallest prime dividing \( n \)?
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What if $p$ is the \textit{smallest} prime dividing $n$?
$\text{ord}_p(2)$ must also divide $n$. 
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$\text{ord}_p(2)$ must also divide $n$.

However, $1 < \text{ord}_p(2) \leq p-1$.

This means that $n$ must have another factor smaller than $p$. 
Observation 2

What if p is the **smallest** prime dividing n? 
ord<p>(2) must also divide n. 
However, 1 < ord<p>(2) ≤ p-1. 
This means that n must have another factor smaller than p. 

Contradiction!