Math 96:
Homework 6

Fall 2024

This homework is due in class on Friday, November 8th. Please complete at least one problem below.

2009 A3: Let d_n be the determinant of the $n \times n$ matrix whose entries, from left to right and then from top to bottom, are $\cos 1, \cos 2, ..., \cos n^2$. (For example,

$$d_3 = \begin{vmatrix} \cos 1 & \cos 2 & \cos 3 \\ \cos 4 & \cos 5 & \cos 6 \\ \cos 7 & \cos 8 & \cos 9 \end{vmatrix}.$$

(The argument of cos is always in radians, not degrees.) Evaluate $\lim_{n\to\infty} d_n$.

1954 A5: If $f(x)$ is a real-valued function defined for $0 < x < 1$, then the formula $f(x) = o(x)$ is an abbreviation for the statement that

$$\frac{f(x)}{x} \to 0 \text{ as } x \to 0.$$

Keeping this in mind, prove the following: if

$$\lim_{x\to 0} f(x) = 0 \text{ and } f(x) - f\left(\frac{x}{2}\right) = o(x),$$

then $f(x) = o(x)$.

1939 A7ii: If the expansion in powers of x of the function

$$\frac{1}{(1-ax)(1-bx)}$$

is given by

$$c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots ,$$

prove that the expansion in powers of x of the function

$$\frac{1+abx}{(1-abx)(1-a^2x)(1-b^2x)}$$

is given by

$$c_0^2 + c_1^2 x + c_2^2 x^2 + c_3^2 x^3 + \cdots .$$

1974 B6: For a set with n elements, how many subsets are there whose cardinality (the number of elements in the subset) is respectively $\equiv 0 \pmod 3, \equiv 1 \pmod 3, \equiv 2 \pmod 3$? In other words, calculate

$$S_{i,n} = \sum_{k \equiv i \pmod 3} \binom{n}{k} \text{ for } i = 0, 1, 2.$$