Math 96:
Combinatorics Techniques

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1 Introduction

Combinatorics is the study of counting things. While this might sound trivial, there are a lot of situations where the objects being considered are too abstract or too large to count directly and so more complicated techniques are required. Combinatorics in general also covers a grab bag of other results about discrete objects.

2 Basic Counting Techniques

The most basic ways to count things are to break the counting problem you have into simpler parts.

The addition rule says that if you can split the thing you are trying to count into disjoint cases, then you can take the sum of the number of things in the individual cases. In particular, if A and B are disjoint sets, then $|A \cup B| = |A| + |B|$.

The multiplication rule says that if an object is constructed by making a pair of choices the total number of ways to make both choices is the number of ways to make the first choice times the number of ways to make the second. In particular, you have $|A \times B| = |A||B|$. More generally, if you make the two choices sequentially, this still holds even if the exact options available for the second choice depend on your first choice so long as the number of possibilities doesn't change.

One slightly less obvious counting formula is that for binomial coefficients. The binomial coefficient $\binom{n}{k}$ ("n choose k") counts the number of ways to select a subset of size k from a specific set of size n. There are several important facts

to know about these coefficients:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$
$$\binom{n}{k} = \binom{n}{n-k}$$
$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$$
$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

These rules can often be repeated many times and applied in careful order in order to count more complicated objects. Furthermore, being able to accurately count things will often have surprising side benefits.

1996 B1: Define a *selfish* set to be a set which has its own cardinality (number of elements) as an element. Find, with proof, the number of subsets of $\{1, 2, \ldots, n\}$ which are *minimal* selfish sets, that is, selfish sets none of whose proper subsets is selfish.

3 Counting Things in Two Different Ways and Linearity of Expectation

One of the most useful tools in combinatorics is counting the same object in two different ways. This gives two formulas for the same number, and thus can be used to prove a number of useful equations. In fact, essentially all of the facts about binomial coefficients above can be proved this way.

Another version of this is what is known as *linearity of expectation*. This says that if you have some quantity given as a $X = A + B$ then the average of X is the same as the average of A plus the average of B. So for example, if a students grade in a class in based on their total score over several tests, the average total score in the class is the sum of the average scores on each of the tests. This is a very versatile technique, especially when you write things in terms of indicator functions (that is a quantity that is 1 if some event happens and 0 if it does not).

2016 B4: Let A be a $2n \times 2n$ matrix with entries chosen independently at random. Every entry is chosen to be 0 or 1, each with probability $1/2$. Find the expected value of $\det(A - A')$ (as a function of n), where A' is the transpose of A.

4 The Pigeonhole Principle

The Pigeonhole Principle states that if more than n pigeons are placed into n holes (with each pigeon being assigned one hole), then some hole must have at

least two pigeons in it. While this seems obvious, the fact that it applies so generally gives a number of important and non-obvious applications. This can often by applied geometrically to packing problems. Noting that if you have many points two of them must lie in the same small box.

You can also consider situations more generally when there are too few or too many of something to go around.

1956 A2: Prove that every positive integer has a multiple whose decimal representation involves all ten decimal digits.

5 Recurrence Relations

Another very useful counting technique is the use of recurrence relations. This often applies if the problem you are working on comes in various sizes and you can relate the answers to the various sized problems to each other. In particular, suppose that the answer to the size n version of your problem is A_n, you would like to be able to relate A_n to values of A_m for $m < n$. If you can do this, at the very least this provides you with a method for computing the values of A_n without too much difficulty. It also means that if you can guess the correct formula for A_n, you can often prove it by induction. Finally, finding such a recurrence will often simply allow you to solve for A_n.

One relatively common case of this is what are known as *linear, homogeneous recurrence relations with constant coefficients*. These occur when you have a recurrence relation of the form:

$$A_n = c_1 A_{n-1} + c_2 A_{n-2} + \ldots + c_m A_{n-m}$$

for some explicit numbers $c_1, \ldots, c_m$. In this case, it turns out that (given some of the initial values) you can solve for A_n exactly. In particular, one can show that A_n must be of the form:

$$A_n = \sum_{\lambda,k} \alpha_{\lambda,k} n^{k-1} \lambda^n$$

where $\alpha_{\lambda,k}$ are some complex numbers (that you will need to solve for) and the sum is over all pairs of complex numbers λ and positive integers k where λ is a root of degree at least k of the *characteristic polynomial*

$$P(x) := x^m - c_1 x^{m-1} - c_2 x^{m-2} - \ldots - c_m.$$

If you need to prove this on the exam, you can find $\alpha_{\lambda,k}$ so that this formula holds for $n = 1, 2, 3, \ldots, m$ and then use induction to prove that it holds for all larger values of n.

1996 B5: Given a finite string S of symbols X and O, we write $\Delta(S)$ for the number of X's in S minus the number of O's. For example, $\Delta(XOOXOOX) = -1$. We call a string S *balanced* if every substring T of (consecutive symbols of) S has $-2 \leq \Delta(T) \leq 2$. Thus, $XOOXOOX$ is not balanced since it contains the substring $OOXOO$. Find, with proof, the number of balanced string of length n.