Math 96:
Combinatorics Techniques

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1 Introduction

Combinatorics is the study of counting things. While this might sound trivial, there are a lot of situations where the objects being considered are too abstract or too large to count directly and so more complicated techniques are required. Combinatorics in general also covers a grab bag of other results about discrete objects.

2 Basic Counting Techniques

The most basic ways to count things are to break the counting problem you have into simpler parts.

The addition rule says that if you can split the thing you are trying to count into disjoint cases, then you can take the sum of the number of things in the individual cases. In particular, if A and B are disjoint sets, then $|A \cup B| = |A| + |B|$.

The multiplication rule says that if an object is constructed by making a pair of choices the total number of ways to make both choices is the number of ways to make the first choice times the number of ways to make the second. In particular, you have $|A \times B| = |A||B|$. More generally, if you make the two choices sequentially, this still holds even if the exact options available for the second choice depend on your first choice so long as the number of possibilities doesn't change.

These rules can often be repeated many times and applied in careful order in order to count more complicated objects. Furthermore, being able to accurately count things will often have surprising side benefits.

1959 B6: Prove that, if x and y are positive irrationals such that $1/x+1/y = 1$, then the sequences $[x], [2x], \ldots, [nx], \ldots$ and $[y], [2y], \ldots, [ny], \ldots$ together include every positive integer exactly once. (The notation $[x]$ means the largest integer not exceeding x.)

3 The Pigeonhole Principle

The Pigeonhole Principle states that if more than n pigeons are placed into n holes (with each pigeon being assigned one hole), then some hole must have at least two pigeons in it. While this seems obvious, the fact that it applies so generally gives a number of important and non-obvious applications. This can often by applied geometrically to packing problems. Noting that if you have many points two of them must lie in the same small box. Another geometric application is the classic result below.

1979 B5 (modified): In the plane, let C be a closed convex set that contains $(0, 0)$ but no other point with integer coordinates. Suppose that C is preserved by reflection about the origin. Prove $A(C) \leq 4$.

4 Binomial Coefficients

One slightly less obvious counting formula is that for binomial coefficients. The binomial coefficient $\binom{n}{k}$ ("n choose k") counts the number of ways to select a subset of size k from a specific set of size n. There are several important facts to know about these coefficients:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$
$$\binom{n}{k} = \binom{n}{n-k}$$
$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$$
$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

2004 B2: Let m and n be integers show that

$$\frac{(m+n)!}{(m+n)^{m+n}} \leq \frac{m!}{m^m} \frac{n!}{n^n}.$$

5 Functional Equations

OK. Not really combinatorics per se, but functional equations are equations where you try to solve for a function based on certain relations between its values. Functional equation problems are a mixed bag and can vary from fairly easy to very difficult. One of the main techniques for functional equations problems are to find convenient values of the variables to plug in in order to get relatively simple relations to show up among your function values. Using these you can hope to solve for selected function values or prove useful relationships.

From there you will often need to build up more powerful lemmas until you solve the full problem.

1959 A3: Find all complex-valued functions f of a complex variable such that $f(z) + zf(1-z) = 1 + z$ for all z.