Math 96:
Calculus Techniques

November 1st, 2024

1 Introduction

Calculus problems are fairly common on the Putnam. While you all should have had at least some exposure to calculus already, there are some tricks and ideas that are worth keeping in mind.

2 Convergent Sums

One important topic is that of showing convergence of sums. One of the most common ways for this to happen is for the sum in question to be absolutely convergent, that is for the sum of the absolute values of the terms of the sum to be convergent. If as sum is absolutely convergent, it and any rearrangement of its terms will be convergent and will always sum to the same value. Showing absolute convergence is usually about estimating how quickly the terms go to 0. If the terms of the sum are decreasing, then the power test says that it will converge if the n^{th} term is much smaller than $1/(n^{1+c})$ for any positive c, but will diverge if the terms are larger than $1/n$. If this fails, there are slightly more refined versions comparing the terms to $1/(n\log(n))$ instead. If the terms of the sum do not come in decreasing order, you can always reorder them so that they do. Then the question of convergence becomes one about how many relatively large terms are in your sum.

For sums that are not absolutely convergent, but may be conditionally convergent, their convergence will depend on cancellation between positive an negative terms in the sum. Perhaps there is a way for example to pair off adjacent positive and negative terms so that the resulting sum is absolutely convergent.

1949 A3: Assume that the complex numbers $a_1, a_2, \ldots, a_n, \ldots$ are all different from zero, and that $|a_r - a_s| > 1$ for $r \neq s$. Show that the series

$$\sum_{n=1}^{\infty} \frac{1}{a_n^3}$$

converges.

3 Compactness

One of the most important results in analysis is that a continuous function on a compact set (for example a closed and bounded subset of $\mathbb{R}^n$) obtains its maximum and minimum values. This result is quite useful for showing that such functions are bounded and are in fact bounded by some obtainable value. In practice, the set on which the function is defined will not always be compact, but there are often tricks that can be used to compactify it. For example, if the function is homogeneous (i.e. $f(x) = f(cx)$ for any scalar multiple c), it is enough to show that f is bounded on the unit ball instead of all of $\mathbb{R}^n$.

1999 A5: Prove that there is a constant C such that, if $p(x)$ is a polynomial of degree 1999, then

$$|p(0)| \leq C \int_{-1}^{1} |p(x)| dx.$$

4 Generating Functions

We know that most reasonable functions $f(x)$ have a Taylor series $f(x) = a_0 + a_1 x + a_2 x^2 + \ldots$. While thinking of this as an actual sum can be useful, it is often useful to think of it merely as a way of keeping track of infinitely many numbers $a_0, a_1, a_2, \ldots$. By manipulating the function f, we can hope to learn interesting things about the underlying sequence. There are several standard manipulations that are worth knowing:

Standard generating functions:

- $\frac{1}{1-cx} = 1 + cx + c^2x^2 + c^3x^3 + \ldots$.
- $e^{cx} = 1 + cx + cx^2/2 + cx^3/3! + ldots.$
- $(1+x)^a = 1 + \binom{a}{1}x + \binom{a}{2}x^2 + \ldots$.

There are also some standard ways to manipulate them:

- $(a_0 + a_1 x + a_2 x^2 + \ldots) + (b_0 + b_1 x + b_2 x^2 + \ldots) = ((a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + \ldots)$.
- $x(a_0 + a_1 x + a_2 x^2 + \ldots) = a_0 x + a_1 x^2 + a_2 x^3 + \ldots$.
- $(a_0 + a_1 x + a_2 x^2 + \ldots)' = a_1 + 2a_2 x + 3a_3 x^2 + \ldots$.
- There are some very useful tools involved with multiplying generating functions that we do not have time for in this lecture.
- There are ways to extract certain coefficients as we shall see below:

1939 B7i: If

$$\begin{aligned} u &= 1 + \frac{x^3}{3!} + \frac{x^6}{6!} + \cdots, & (1) \\ v &= \frac{x}{1!} + \frac{x^4}{4!} + \frac{x^7}{7!} + \cdots, & (2) \\ w &= \frac{x^2}{2!} + \frac{x^5}{5!} + \frac{x^8}{8!} + \cdots, & (3) \end{aligned}$$

prove that

$$u^3 + v^3 + w^3 - 3uvw = 1.$$

5 Complex Numbers

Complex numbers are numbers of the form $x+iy$ where x and y are real numbers and i is the imaginary unit satisfying $i^2 = -1$. These are useful tools not just for solving polynomials (as in the Fundamental Theorem of Algebra), but also for a number of other things. You can add and multiply complex numbers and can divide by non-zero ones. A complex number has an absolute values $|x + iy| = \sqrt{x^2 + y^2}$ that is multiplicative in the sense that $|z_1 z_2| = |z_1||z_2|$.

Geometrically, you can think of complex numbers as corresponding to points in the $x - y$ plane. The absolute value of a complex number is just the geometric distance from the origin. When you multiply two complex numbers, the distances from the origin multiply and the angles from the positive x-axis add.

It is also worth noting that there is an exponential function defined on complex numbers as

$$e^{x+iy} = e^x \cos(y) + ie^x \sin(y),$$

which has the properties you would expect (like $e^{z_1+z_2} = e^{z_1}e^{z_2}$). It is also very convenient that some of the standard trigonometric functions can be written in terms of these complex exponentials. Namely,

$$\cos(x) = \frac{e^{ix} + e^{-ix}}{2}, \qquad \sin(x) = \frac{e^{ix} - e^{-ix}}{2i}.$$

1989 A3: Prove that if

$$11z^{10} + 10iz^9 + 10iz11 = 0,$$

then $|z| = 1$. (Here z is a complex number and $i^2 = 1$.)