Math 184 Homework 5

Fall 2022

Question 1 (Odd Permutation Recursion, 30 points)**.** *Use a direct combinatorial proof to show that (using the convention that* $\mathrm{ODD}(0) = 1$*) for* $n \geq 1$

$$\mathrm{ODD}(n) = \sum_{\substack{1 \leq k \leq n \\ k \ odd}} (n-1)_{k-1} \mathrm{ODD}(n-k).$$

Note: You can use this recursion to provide an alternative proof of the formula for $\mathrm{ODD}(n)$*.*

Proof. We claim that the k^{th} term on the right hand side is the number of permutations of $[n]$ into odd cycles where the element n is in a cycle of length exactly k. Summing over k odd should then give the total number of permutations of $[n]$ into odd cycles. We count this in the following way: first we determine the cycle that contains n. This should be a cycle of the form $(n\pi(n)\pi^2(n)\dots\pi^{k-1}(n))$. The terms $\pi(n), \pi^2(n), \dots, \pi^{k-1}(n)$ can be any $k-1$ distinct elements of $[n-1]$, so there are $(n-1)_{k-1}$ ways to select them. Once we have selected them, there are $\mathrm{ODD}(n-k)$ ways to define a permutation on the remaining $n-k$ elements, splitting them into odd cycles. This completes our proof. □

Question 2 (Odd Number of Sets Inclusion Exclusion, 30 points)**.** *Let* $A_1, A_2, \dots, A_n$ *be finite sets. Show that the number of elements* x *that are in an* odd *number of the* A_i*'s is given by*

$$\sum_{k=1}^{n} (-2)^{k-1} \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} |A_{i_1} \cap A_{i_2} \cap \cdots A_{i_k}|.$$

Hint: determine how many times each element x *is counted. You may want to use the Binomial Theorem.*

Proof. Consider an element $x \in [n]$ which is in l many sets A_{i_1}, A_{i_2}, ..., A_{i_l}. Let S be this index set $\{i_1, i_2, ..., i_l\}$. We will find how many times x is counted by the sum. Note if x is in $l = 0$ many sets, it won't be counted by the sum at all. Furthermore, the second summation will only count x if the index set of the sum is a subset of S. Thus to see how many times x is counted, we only consider the sum where the second summation is indexed over subsets of S. We then get a summation of the form:

$$(|A_{i_1}| + |A_{i_2}| + \cdots) - 2(|A_{i_1} \cap A_{i_2}| + |A_{i_1} \cap A_{i_3}| + \cdots) + 2^2(|A_{i_2} \cap A_{i_2} \cap A_{i_3}| + \cdots) + \cdots + (-2)^l(|A_{i_1} \cap \cdots \cap A_{i_l}|),$$

which is an alternating sum of k-fold intersections, just as in the inclusion-exclusion formula, but with additional factors of 2 as well. So, we have x is counted l times in the first group of 1-fold intersections, $-2\binom{l}{2}$ times in the second group of 2-fold intersections, and generally $(-2)^{k-1}\binom{l}{k}$ many times in k-fold intersections. Thus we need to calculate the following sum:

$$\sum_{k=1}^{l} (-2)^{k-1} \binom{l}{k} = -\frac{1}{2} \sum_{k=1}^{l} (-2)^k \binom{l}{k} = -\frac{1}{2}\Big(-1 + \sum_{k=0}^{l} (-2)^k \binom{l}{k} \Big) = -\frac{1}{2}(-1 + (1-2)^l) = \frac{1}{2}((-1)^{l+1} + 1)).$$

Note the use of the binomial formula for $\big(1 + (-2)\big)^l = \sum_{k=0}^{l} (-2)^k \binom{l}{k}$. When l is even, this sum is 0, and when l is odd, this sum is 1. Thus the original sum counts each element which is in an odd number of sets exactly once, as claimed.

□

Question 3 (Necklace Counting, 40 points)**.** *You are trying to create a necklace with n beads labelled $1, 2, \ldots, n$ arranged in a circle. Each bead is one of k different colors and adjacent beads must be different colors. Prove that the number of ways to create such a necklace is*

$$(k-1)^n + (-1)^n(k-1).$$

Hint: Use inclusion-exclusion on the conditions that a given pair of adjacent beads are the same color. You may need to use the binomial theorem to simplify your answer.

Proof. There are k^n total possible necklaces, so to get our answer, we need to subtract the number which have some pair of adjacent beads the same color. To count those, we will use inclusion-exclusion on the number of indices i such that bead i and $i+1$ have the same color.

Let A_i be the set of necklaces where bead i and $i-1$ are the same color (A_1 is the set where bead n and 1 are the same color). By inclusion-exclusion, the total number of necklaces with at least one pair of adjacent beads the same color is

$$\sum_{j=1}^{n}(-1)^{j+1}\sum_{1\le i_1<i_2<\ldots<i_j\le n}|A_{i_1}\cap A_{i_2}\cap\ldots\cap A_{i_j}|$$

Now we need to compute the sizes of each of those intersections. We first look at one special case, where $j=n$. In that case all n beads must be the same color, so there are k options.

Suppose we're looking at $|A_{i_1}\cap A_{i_2}\cap\ldots\cap A_{i_j}|$. Let $S=\{i_1,i_2,\ldots i_j\}$ be the collection of indices we're intersecting over. We already dealt with the $j=n$ case, so here we have $j<n$, so there is some index $k\notin S$. Rotating the entire necklace doesn't change how many possibilities there are, so we may assume that $1\notin S$ (to make notation easier).

Now we go along the necklace, starting at bead 1, and selecting a valid color for each bead. We have k choices for bead 1. Now suppose we're at some bead r. If $r\in S$, then the r^{th} bead and the $(r-1)^{st}$ bead are the same color, so we must give the r^{th} bead the color of the $(r-1)^{st}$ bead, which means we have only one choice. Otherwise we can pick freely from any of the k choices.

Then if $|S|=j$, we will have j choices with 1 option, and $n-j$ choices with k options, all independent. This tells us that $|A_{i_1}\cap A_{i_2}\cap\ldots\cap A_{i_j}|=k^{n-j}$.

The number of bad necklaces is therefore

$$(-1)^{n+1}k+\sum_{j=1}^{n-1}(-1)^{j+1}\sum_{1\le i_1<i_2<\ldots<i_j\le n}k^{n-j}.$$

For each j there are $\binom{n}{j}$ many choices for $i_1,i_2,\ldots i_j$ (pick any j elements and their order is forced), so we get

$$(-1)^{n+1}k+\sum_{j=1}^{n-1}(-1)^{j+1}\binom{n}{j}k^{n-j}.$$

Our final answer will be k^n minus this expression - let's do that subtraction now, because it will make the computation easier. The number of valid necklaces is (subtracting is equivalent to removing a power of (-1))

$$k^n+(-1)^nk+\sum_{j=1}^{n-1}(-1)^{j}\binom{n}{j}k^{n-j}.$$

This almost looks like something we can apply binomial theorem to, but there are a couple small things missing: we need the sum to run from 0 to n rather than 1 to $n-1$, and we should pull out the factor of k^n. We can solve one problem using the k^n term at the front: $k^n=(-1)^0\binom{n}{0}k^{n-0}$, so it is precisely the missing $j=0$ term in our sum.

As for the $j=n$ term, we'll do the classic trick of adding 0: the $j=n$ term should be $(-1)^n\binom{n}{n}k^0=(-1)^n$, so we just add and subtract $(-1)^n$. This tells us that the number of valid necklaces is

$$
\begin{aligned}
&k^n + (-1)^n k + \sum_{j=1}^{n-1} (-1)^j \binom{n}{j} k^{n-j} \\
=&(-1)^n k + \sum_{j=0}^{n-1} (-1)^j \binom{n}{j} k^{n-j} \\
=&(-1)^n k + \left(\sum_{j=0}^{n-1} (-1)^j \binom{n}{j} k^{n-j} \right) + (-1)^n - (-1)^n \\
=&(-1)^n k + \left(\sum_{j=0}^{n} (-1)^j \binom{n}{j} k^{n-j} \right) - (-1)^n \\
=&(-1)^n (k-1) + \left(\sum_{j=0}^{n} (-1)^j \binom{n}{j} k^{n-j} \right) \\
=&(-1)^n (k-1) + k^n \left(\sum_{j=0}^{n} (-1)^j \binom{n}{j} k^{-j} \right) \\
=&(-1)^n (k-1) + k^n \left(\sum_{j=0}^{n} \binom{n}{j} \left(\frac{-1}{k} \right)^j \right) \\
=&(-1)^n (k-1) + k^n \left(1 + \frac{-1}{k} \right)^n \\
=&(-1)^n (k-1) + k^n \left(\frac{k-1}{k} \right)^n \\
=&(-1)^n (k-1) + (k-1)^n
\end{aligned}
$$

which is what we wanted to show. □

Question 4 (Extra credit, 1 point)**.** *Approximately how much time did you spend working on this homework?*