Math 184 Homework 4

Fall 2022

Question 1 (Distinct Parts Partitions, 20 points)**.** *Show that the number of partitions of n into exactly k distinct parts is*

$$p_k(n - k(k-1)/2).$$

Proof. Let $\mathcal{A}$ be the set of partitions of n into k distinct pieces, and $\mathcal{B}$ be the set of partitions of $n-k(k-1)/2$ into k pieces. We will construct a bijection $f : \mathcal{B} \to \mathcal{A}$ as follows: if we have a partition of $n-k(k-1)/2$ into k pieces, write them in descending order, so $n-k(k-1)/2 = x_1+x_2+\dots x_k$, where $x_1 \geq x_2 \geq x_3 \dots \geq x_k$. Now we will build a partition of n as follows: let $y_1 = x_1+(k-1)$, $y_2 = x_2+(k-2)$, $y_3 = x_3+(k-3), \dots y_k = x_k$. Then since $x_i \geq x_{i+1}$, we have that $y_i > y_{i+1}$, so the sequence of ys is strictly decreasing, hence they're distinct. We also have $y_1 + y_2 + \dots y_k = x_1 + (k-1) + x_2 + (k-2) + \dots x_{k-1} + 1 + x_k = x_1 + x_2 + \dots x_k + (k-1) + (k-2) + \dots 1 = n - k(k-1)/2 + k(k-1)/2 = n$, so this does partition n into k distinct pieces.

Now we need to show that f is a bijection.

If we have two different partitions $x_1 + x_2 + \dots x_k$ and $z_1 + z_2 + \dots z_k$, there must be some i so that $x_i \neq z_i$ (otherwise they would be the same partition). Then $x_i + (k-i) \neq z_i + (k-i)$, so they map to different partitions under f.

To see it is surjective, if we have a partition of n into k distinct pieces, we can order them in descending order and subtract $k-i$ from the i^{th} piece. This will give us an element of $\mathcal{A}$ which maps to our partition under f.

Thus f is a bijection, so the two sets have the same size. □

Question 2 (Permutation Orders, 20 points)**.** *Let π be a permutation of $[n]$. Show that $\pi^{n!}$ is the identity permutation (i.e. the permutation σ of $[n]$ where $\sigma(i) = i$ for all $1 \leq i \leq n$).*

Proof. For any permutation π of $[n]$, there is an $i \in [n]$ such that $\pi^i(x) = x$. Note that we can compose $\pi^{n!}$ as $((((\pi \circ \pi) \circ \pi) \cdots \circ \pi) \circ \pi)$ or as $(\pi^i)^{n(n-1)\cdots(i+1)(i-1)\cdots 1}$ (assuming $i \neq 1, n$, in which case the exponent is $i(n(n-1)\cdots 2)$ or $i(n-1)!$). The latter case makes it clearer that we have $(\pi^i)^{n!/i}(x) = x$, and since we can do this for every element, we have $\pi^{n!}$ is the identity permutation. □

Question 3 (Ambiguous Permutations, 30 points)**.** *We have discussed two ways to encode a permutation π of $[n]$ as an ordered list of the numbers from 1 to n: either by listing the values $\pi(1), \pi(2), \dots, \pi(n)$ in order or by writing π in canonical cycle notation and removing the parentheses. Call a permutation π* ambiguous *if each of these representations gives rise to the same sequence of numbers. Show that the number of ambiguous permutations of $[n]$ is given by the Fibonacci number F_n where $F_1 = 1, F_2 = 2$, and $F_n = F_{n-1} + F_{n-2}$ for all $n > 2$.*

Proof. In Canonical Cycle Notation (CCN), the last cycle must start with n, and it must end in $\pi(n)$ in order to be ambiguous (recall that an ambiguous permutation can be listed as $\pi(1), \pi(2), ..., \pi(n)$ in CCN). The element after n in CCN is what π maps n to, or in other notation, it is $\pi(n)$. This means the last cycle must have length one or two: we can't have any other elements between n and $\pi(n)$, as $\pi(n)$ both follows n and ends the cycle, but we don't exclude the possibility that n maps to itself.

If it is length one, we have $\pi(n) = n$, and our permutation in CCN ends with (n). Then the number of ambiguous permutations of $[n]$ ending in (n) is the same as the number of ambiguous permutations on $[n-1]$.

Suppose the last cycle is length two. To be ambiguous, the second to last entry (which is n) must be equal to $\pi(n-1)$. So $n-1$ is the last entry in the cycle containing n in CCN (because the last element

of a cycle maps to the element starting the cycle and $\pi(n-1) = n$), but this last cycle is only length two. So far our permutation is $_, _, _, (n, n-1)$. Then our next to last cycle must start with the largest unused number, $n-2$, in order to be in CCN and must end in $\pi(n-2)$ to be ambiguous. This is the same pattern we started out with, so if the last cycle is length two, then the number of ambiguous permutations on $[n]$ ending in $(n, n-1)$ is the same as the number of ambiguous permutations on $[n-2]$.

This tells us that we have the same recursion as the Fibonacci sequence: if A_n is the number of ambiguous permutations on $[n]$, we have shown $A_n = A_{n-1} + A_{n-2}$ for $n > 2$. To show that it is in fact the same sequence, we must confirm that they have the same initial conditions as well. $A_1 = 1$ because we have one permutation on [1], and it is ambiguous, and $A_2 = 2$ because we have two permutations, 12 and 21 in one line notation and (1)(2) and (21) in CCN, which are both ambiguous. □

Question 4 (Random Permutations and Coin Flips, 30 points)**.** *Consider the following two random processes:*

1. *Pick a uniform random permutation π of $[n]$ and count the number of cycles in π.*
2. *Flip n coins $C_1, C_2, \ldots, C_n$ where C_i has a probability $1/i$ of coming up heads, independently of the other coins and count the total number of heads.*

Show that these processes give rise to the same distribution over outcomes.

Proof. We will show this by induction on n. When $n = 1$, there is only one permutation, which has one cycle, and one coin that always comes up heads.

Now let $n \geq 1$, and suppose that for $n-1$, the number of cycles in a random permutation and the coin process give the same distribution.

Let P_n be the number of cycles in a random permutation of $[n]$, and H_n be the number of heads we generate from flipping the coins $C_1, C_2 \ldots C_n$. We want to show that P_n and H_n have the same distribution.

There are $c(n,k)$ permutations with k cycles, so the probability that $P_n = k$ is $c(n,k)/n!$ since the distribution is uniform. Using the recurrence for Stirling numbers, this is equal to

$$\begin{aligned}&\frac{c(n-1,k-1)}{n!} + \frac{(n-1)c(n-1,k)}{n!}\\ =&\frac{1}{n}\frac{c(n-1,k-1)}{(n-1)!} + \frac{n-1}{n}\frac{c(n-1,k)}{(n-1)!}\end{aligned}$$

Now $c(n-1,k-1)$ is the number of permutations of $[n-1]$ with k cycles, so by induction $c(n-1,k-1)/(n-1)!$ is the probability we get $k-1$ heads from $C_1, C_2, \ldots C_{n-1}$. Also by induction, $c(n-1,k)/(n-1)!$ is the probability we get k heads from those coin flips.

Now $H_n = k$ exactly when we get k heads from $C_1, C_2, \ldots C_n$. To do this, we can either have C_n be heads and get $k-1$ heads from the first $n-1$, or C_n is tails and we get k heads from the first $n-1$. The probability this happens is

$$\frac{1}{n}\frac{c(n-1,k-1)}{(n-1)!} + \frac{n-1}{n}\frac{c(n-1,k)}{(n-1)!}$$

which is precisely the probability that a random permutation of $[n]$ has k cycles. Thus P_n and H_n have the same distribution, which completes the induction.

Alternative counting solution: We will generate the canonical form of a random permutation in n steps as follows:

Step 1: Write down n.

Step 2: Roll a 2-sided die. On a 1, place $n-1$ to the left of n, and on a 1 place it to the right.

Step 3: Roll a 3-sided die. If the roll is r, write down $n-2$ so it's the r^{th} number on the list (ie, if you roll a 1, $n-2$ will be the first number)

$\vdots$

Step i: Roll an i-sided die. If the result is r, write down $n - i + 1$ so it's the r^{th} number on the list.

$$\vdots$$

Step n: Roll an n-sided die. If the result is r, write down 1 so it's the r^{th} number on the list.
Example procedure: if $n = 5$, and we get rolls of $2, 2, 1, 4$, the sequence we generate would be

$$\begin{aligned} &5 \\ &5\ 4 \\ &5\ 3\ 4 \\ &2\ 5\ 3\ 4 \\ &2\ 5\ 3\ 1\ 4 \end{aligned}$$

This will generate a uniformly random permutation of $[n]$.[1]

At each step, the only way the number we place down can be in a new cycle is if we roll a 1, since it must come before all larger numbers. This means at step i of the process, we add a new cycle with probability $1/i$, which shows that the number of cycles has the same distribution as the coin flips.

□

[1]To argue that what we get is a uniformly random permutation, note that because our dice rolls were independent, each of the $n!$ possible sets of dice rolls are equally likely. If two sets of rolls differ, look at the first step in which you rolled different numbers. The number you place in this step will have swapped sides of some already placed number, so will generate you different permutations.