Math 184 Homework 3

Fall 2022

Question 1 (Dice Designs, 35 points)**.** *When making a die one must put the numbers* 1 *through* 6 *on the six faces of a cube so that each number appears on a distinct face. We say that two such designs are the same if it is possible to rotate one of the cubes so that the numbers on its faces agrees with the others'. How many different designs are there?*

Hint: Relate the number of designs to the number of ways of labelling six faces 1 *through* 6*. Not every labelling is a distinct design, but can you quantify the difference?*

Proof. Choose your favorite number from 1 to 6 and fix that face with your favorite number as the top of the die. If we did not fix this face, we could look at the same design from different perspectives and see different patterns, so we would overcount by not considering the rotations. Four of the remaining five numbers must be on the faces adjacent to the top, so we have $\binom{5}{4} = 5$ choices of numbers for these four faces. When we consider the orders they could be in, fix a face to be the first because otherwise we could rotate around the axis through the top and bottom of the die and see that different patterns yield the same design. Thus we have 3! orders for these four faces. Given these choices, we only have one option left for the bottom face, so we have $5 \cdot 3! = 30$ distinct designs.

Alternatively, consider the number of designs in relation to the number of labellings of the faces as suggested in the hint. There are 6! different ways of numbering six faces. But since each die can be rotated in 24 different ways, each die design accounts for 24 of these numberings. Thus the answer is 6!/24.

Question 2 (Composition Counting, 30 points)**.** .

(a) *How many compositions of n into k parts of size* 1 *and* 2 *are there? [10 points]*

(b) *How many compositions of n into any number of parts with one of the parts colored red are there? [10 points]*

(c) *What is the sum over compositions of n into k parts of the product of the sizes of the parts? Hint: Count the number of stars and bars configurations with one star in each part highlighted. [10 points]*

Proof. (a) If we want to write n as the sum of k 1s and 2s, we will have to use i 2s and $k-i$ 1s for some $0 \leq i \leq k$. This will give us a sum of $2i+(k-i) = i+k$, so $n = i+k$, or $i = n-k$. Thus we need $n-k$ 2s and $k-(n-k) = 2k-n$ 1s. These must both be non-negative, so this is only possible if $k \leq n \leq 2k$. In that case, to define the composition we simply have to choose which $n-k$ elements of the composition are 2, which can be done in $\binom{k}{n-k}$ ways.

Alternate explanation using balls and bins: Since we need to break n into a (not weak) composition of k parts, we can first put one ball into each of k bins. Next, we need to decide some of the bins to get a second ball so that we have a total of n balls. Thus we have a choice of $n-k$ of the k bins to have a second ball, giving us $\binom{k}{n-k}$.

(b) We will give two proofs, one a clean combinatorial argument, and the other a slightly more brute-force approach.

Proof 1: We know that we can get compositions of n using stars and bars. We then need to pick which piece to color red. We will do that by coloring the bar before the chosen piece red, or no red bar if we wish to color the first piece red. Thus what we're looking to count is equal to the number of stars and bars with no empty bins, and 0 or 1 bars colored red.

If there are 0 bars colored red, we can do this in 2^{n-1} ways: line up the n stars, and then each gap between them can either have a bar or not.

If there is 1 bar colored red, we first pick out the red bar, for which we have $n-1$ choices. Then we pick whether or not each other space has a bar, which we can do in 2^{n-2} ways. The total for this case is then $(n-1)2^{n-2}$.

Adding the two cases together, we get a total of

$$2^{n-1} + (n-1)2^{n-2} = 2 \cdot 2^{n-2} + (n-1)2^{n-2} = (n+1)2^{n-2}.$$

Proof 2:

Let's break this problem down by considering how many pieces the composition uses. If it uses k pieces, we start by making the composition. We will use stars and bars. We need each piece to be strictly positive, so we begin by assigning one star to each bin. Then we have $n-k$ stars and $k-1$ bars, so there are $\binom{n-k+(k-1)}{k-1} = \binom{n-1}{k-1}$ configurations. Then we need to pick which of the k parts to color red, so we get a total of $k\binom{n-1}{k-1}$. The total number of compositions with one part colored red is

$$\sum_{k=1}^{n} k\binom{n-1}{k-1}.$$

We wish to evaluate the summation $\sum_{k=1}^{n} k\binom{n-1}{k-1}$. Recall that we have an identity for a similar expression: $\sum_{k=1}^{n} k\binom{n}{k} = n2^{n-1}$. We will use re-indexing, this identity, and some algebraic manipulation to break down the summation.

$$\begin{aligned}
\sum_{k=1}^{n} k\binom{n-1}{k-1} &= \sum_{k=0}^{n-1}(k+1)\binom{n-1}{k} \\
&= \sum_{k=0}^{n-1}\left(k\binom{n-1}{k} + \binom{n-1}{k}\right) \\
&= \sum_{k=0}^{n-1} k\binom{n-1}{k} + \sum_{k=0}^{n-1}\binom{n-1}{k} \\
&= \sum_{k=0}^{n-1} k\binom{n-1}{k} + \sum_{k=0}^{n-1}\binom{n-1}{k}
\end{aligned}$$

Note here that the $k=0$ term in the first sum is 0, so we can begin at $k=1$ with no change.

$$\begin{aligned}
&= (n-1)2^{(n-1)-1} + 2^{n-1} \\
&= (n+1)2^{n-2}
\end{aligned}$$

(c) First, as per the hint, note that if we take a composition of n as $p_1 + p_2 + \ldots + p_k$ and imagine this as putting stars into bins, the total number of ways we can highlight one star from each bin is $p_1 p_2 \ldots p_k$, which are precisely the terms we want to add up. So adding up all of them is equivalent to counting up the number of stars and bars configurations with one star from each bin highlighted.

How do we count these? Let's consider a bin that has some number of balls in it:

$$|\star\star\star\star\star\star\star|$$

We can highlight one of them

$$|\star\star\star\star\star\star\star|$$

We could specify the highlighted star by picking how many stars come before and after it



...which looks suspiciously like another stars and bars problem. For this one we've lost one star per bin (that has to go in the middle), but gained one bar per bin, because we split each bin into 2. Note that we now have a weak composition as highlighting the first star in a bin gives us two adjacent bars and an empty bin. So we have $n-k$ stars and $2k-1$ bars, so the total number of such configurations is $\binom{n+k-1}{2k-1}$

□

Question 3 (Set Partition Identity, 35 points)**.** *Prove that for any positive integer n that:*

$$\sum_{k=1}^{n} 2^k S(n,k) = \sum_{m=0}^{n} \binom{n}{m} B(n-m)B(m).$$

Hint: Show both sides equal the number of set partitions of $[n]$ where each part in the partition is colored red or blue.

Proof. For a given partition of n into k parts, the total number of ways to color the parts either red or blue is 2^k. Thus the left hand side of the equation counts the number of partitions of n into any size $1 \leq k \leq n$ with parts colored red or blue. Now for the right hand side, consider the elements of $[n]$. Color some of the elements blue, say m many of them. Then we have $\binom{n}{m}$ choices of elements to color blue and $B(m)$ ways to partition these m elements into any number of parts. Then the remaining $n-m$ elements are red, and we have $B(n-m)$ ways to partition them into any number of parts. Recall that the order of these parts does not matter for a partition as it does in a composition, so by summing over all values of $0 \leq m \leq n$, the right hand side also counts the number of partitions of n into any number of parts with each part colored red or blue.

Question 4 (Extra credit, 1 point)**.** *Approximately how much time did you spend working on this homework?*