Math 184 Homework 2 Solutions

Fall 2022

Question 1 (Other Pigeonhole Principle Generalizations, 30 points)**.** *Suppose that each of n pigeons are assigned to one of m holes. For which values of m and n can each of the following be guaranteed?*

(a) There are at least n (possibly overlapping) pairs of pigeons that share the same hole. [10 points]

(b) There are at least $n/2$ pigeons that share a hole with some other pigeon. [10 points]

(c) There are at least 2 holes each with at least 2 pigeons. [10 points]

Proof. (a) First consider two cases. If every one of the m holes has three pigeons for a total of $n = 3m$, then each hole has $\binom{3}{2} = 3$ pairs of pigeons sharing a hole for a total of $3m \geq n$ pairs of pigeons. If we have $3m - 1$ pigeons, by distributing the pigeons as evenly as possible amongst the holes we must have one hole with two pigeons and no hole with more than three pigeons, giving $3(m-1) + 1 < 3m - 1$ pairs. We will generalize to show that $n \geq 3m$ suffices and that when $n < 3m$, we can distribute the pigeons such that there are fewer than n pairs.

We claim this "evenly as possible" distribution is the worst case for the number of pairs of pigeons sharing a hole. If $n \geq 3m$, write $n = am + b$, $0 \leq b < m$. If $b = 0$, then we have a pigeons in every hole. If we move a pigeon from one hole to another to make it less even, we lose $\binom{a}{2} - \binom{a-1}{2} = a - 1$ pairs and we gain $\binom{a+1}{2} - \binom{a}{2} = a$, and so we have a net gain of one pair. If $b \neq 0$, then we have a pigeons in $m - b$ holes and $a + 1$ pigeons in b holes. Again, we will move a pigeon and show a net gain in number of pairs. Note that taking a pigeon from a hole with $a + 1$ and moving it to a hole that had a pigeons is equivalent to the starting state. Thus, do the reverse. We gain $\binom{a+2}{2} - \binom{a+1}{2} = a + 1$ pairs and lose $\binom{a+1}{2} - \binom{a}{2} = a$, again gaining a net one pair. Any move we make from the even distribution increases the number of pairs of pigeons that share the same hole.

In addition, if we are in a less even state, any move we make towards a more even state decreases the number of pairs. To make a state more even, we could take a pigeon from a hole containing $a + 2 + j$ pigeons and move it to a hole containing at most $a - 1 - k$ pigeons, for $j, k \geq 0$. This gives us $\binom{a-1-j+1}{2} + \binom{a+2+k-1}{2} - (\binom{a-1-j}{2} + \binom{a+2+k}{2}) = a - 1 - j - (a + 2 + k - 1) = -k - j - 2$ many pairs, where $j \geq 0,\ k \geq 0$, which is always a net loss. Thus any move away from the even distribution increases the number of pairs and any move towards the even distribution decreases the number of pairs.

Now suppose for $n < 3m$ that there are p_i pigeons in the ith hole. The total number of pairs of pigeons sharing a hole minus n is $\sum_{i=1}^{m} (\binom{p_i}{2} - p_i)$. (Why do we consider this quantity? This is non-negative precisely when there are at least n pairs of pigeons sharing a hole). When we distribute the pigeons as evenly as possible, there are at most 3 pigeons in each hole and at least one hole with fewer than three pigeons, and we see that $\sum \binom{p_i}{2} - p_i < 0$ (note if $p_i = 0, 1$ then there are 0 pairs and we avoid the binomial coefficient). This concludes that we cannot guarantee n pairs for $n < 3m$.

For every additional pigeon after the first $3m$, assuming the worst case "evenly as possible" distribution, we must place the new pigeon in some hole with at least 3 pigeons, creating at least 3 new pairs for every new pigeon. Thus for $n \geq 3m$, we always have at least n pairs of pigeons sharing a hole.

(b) If we have $n \leq m$, then we can distribute at most one pigeon per hole and have no pigeons sharing a hole with some other pigeon. Consider $n > m$, then there are at most $m - 1$ pigeons by themselves since by pigeonhole principle we have some hole with at least two pigeons. Thus we need $n - (m - 1)$, the

number of pigeons together in the last hole, to be at least $\frac{n}{2}$. Solving this inequality, we see $n \geq 2(m-1)$ is sufficient to guarantee at least $\frac{n}{2}$ pigeons share a hole with some other pigeon.

No smaller bound can work, because if $n < 2(m-1)$, the configuration which has $m-1$ isolated pigeons and the other $n-(m-1)$ in the same hole has strictly fewer than $n/2$ pigeons sharing a hole with another pigeon.

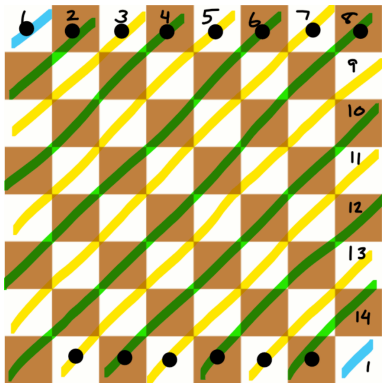
(c) We cannot guarantee this for any n and m. For any values $n, m \geq 1$, simply map all pigeons into one hole.

□

Question 2 (Bishop Packing, 20 points)**.** *In chess a bishop can move diagonally any number of squares. Prove that if* 15 *bishops are placed on an* 8×8 *chessboard that at least two will be attacking each other.*

Hint: Try to divide the board into 14 *regions so that any two bishops in the same region would attack each other.*

Proof. Consider the 13 diagonals of the chessboard with length greater than one. Each of these will be a region (highlighted green or yellow in the diagram). The two remaining squares will be the remaining region (highlighted in blue). Suppose for contradiction there were 15 non-attacking bishops placed on the board. By pigeonhole principle, at least two must lie in the same region. However, since each region lies within some diagonal of the board (the two blue squares are on a main diagonal in the opposite direction of the other regions), the bishops can attack each other within the regions. Thus we have two attacking bishops.



The 14 dots on the diagram above show one possible arrangement of 14 bishops which do not attack each other, so the bound is tight.

Question 3 (Counting Subsets, 50 points)**.** *In the following* $[n]$ *will be used to denote the set* $\{1, 2, 3, \ldots, n\}$*. For each of the following, give a formula for the number of sets of the type specified along with the actual number.*

(a) How many subsets of $[10]$ *are there? [5 points]*

(b) How many subsets of $[10]$ *are there of size exactly* 5*? [5 points]*

(c) How many subsets of $[12]$ *consist of exactly* 4 *even numbers and* 3 *odd numbers? [5 points]*

(d) How many subsets of $[12]$ *have the same number of even and odd numbers in them? [5 points]*

(e) How many subsets of $[20]$ *contain exactly one element with each last digit? [5 points]*

(f) How many subsets of $[20]$ *contain at least one element with each last digit? [5 points]*

(g) How many subsets of $[10]$ *contain exactly one multiple of* 3 *and no other elements within* 1 *of that multiple of* 3*? [5 points]*

(h) How many subsets of $[20]$ *have largest and smallest elements that differ by at most* 5*? [5 points]*

(i) How many subsets of $[10]$ *contain no pair of consecutive numbers? Hint: Figure out the number of such subsets of* $[n]$ *for smaller values of* n *as well. [5 points]*

(j) How many subsets of $[12]$ *contain no pair of elements where one is exactly twice the other? [5 points]*

Proof. (a) To pick a subset of [10], we need to decide for each element $1, 2, \ldots, 10$ whether we include it or not. We have 2 choices per element, and 10 *independent* choices to make, so the total number of subsets is $2^{10} = 1024$

(b) We can pick a subset of size 5 in $\binom{10}{5} = 252$ ways, by the definition of binomial coefficients.

(c) We need to pick 4 of 6 even numbers, and 3 of 6 odd numbers. The total number of ways to do this is $\binom{6}{4} \times \binom{6}{3} = 300$.

(d) To get the same number of even and odd numbers, we can have any number between 1 and 6 of each. If we have k of each number, we can pick the even numbers in $\binom{6}{k}$ ways, and the odd numbers in $\binom{6}{k}$ ways, so the total number of possibilities is

$$\sum_{k=0}^{6} \binom{6}{k}^2 = 924.$$

It turns out this is also equal to $\binom{12}{6}$. Is there a natural interpretation of this answer? Indeed there is; we can also construct our set as follows: Choose any set A of 6 numbers between 1 and 12. Now take the even numbers in A, and the odd numbers in A^c. This gives us a bijection (you should verify this is actually a bijection)

$$\left\{ \begin{array}{c} \text{Subsets of 12} \\ \text{of size exactly 6} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Subsets of 12 with the same} \\ \text{number of even and odd numbers} \end{array} \right\}$$

(e) For each last digit, we have two choices: 1 or 11, 2 or 12, up to 10 or 20. Each is independent, and there are 10 choices, so there are $2^{10} = 1024$ such sets.

(f) This is similar to the last case, but now with the additional option of including both numbers with the last digit (ie for the 1 case, we could include 1, 11, or 1 and 11). This gives us 3 options per digit, for a total of $3^{10} = 59049$ such sets.

(g) We have 3 options for the multiple of 3: 3,6, or 9. Then we have to pick the other numbers in the set. There are 10 total numbers: of those, 1 is in the set, 2 are multiples of 3, and 2 are within 1 of our multiple of 3. So there are 10-1-2-2 = 5 numbers left, each of which can either be in the set or not in the set, for $2^5 = 32$ options once we've picked our multiple of 3. This makes a total of $3 \times 32 = 96$ total sets.

(h) We will count these sets by the smallest number they contain. If the smallest number is k, our set could contain any of $\{k+1, k+2, k+3, k+4, k+5\}$, so long as they're at most 20. If $1 \leq k \leq 15$, we can pick any subset of $\{k+1, k+2, k+3, k+4, k+5\}$ to add for 32 total choices. But for $k = 16$ we only have 4 elements for 16 choices, $k = 17$ we have 8 choices, $k = 18$ 4 choices, $k = 19$ 2 choices, and $k = 20$ 1 choice. The total number of these sets then is

$$15 * 32 + 16 + 8 + 4 + 2 + 1 = 511.$$

Note depending on your reading of the question, you could include the empty set, which would give 512 total sets.

(i) Let $\mathcal{S}_n$ be the collection of subsets of $[n]$ which do not contain any consecutive elements. I claim that $|\mathcal{S}_n| = F_{n+2}$, where F_n is the n^{th} Fibonacci number[1]. We will prove this by induction. We have two base cases, $n = 1$ and $n = 2$.

[1]Recall that the Fibonacci sequence is the sequence 1, 1, 2, 3, 5, 8, 13 ... defined recursively as follows: $F_1 = F_2 = 1$, and $F_{n+2} = F_{n+1} + F_n$ for all $n \geq 1$.

When $n = 1$, there are two such subsets: $\emptyset$ and $\{1\}$, so $|\mathcal{S}_1| = 2 = F_3$

When $n = 2$, there are three such subsets: $\emptyset$, $\{1\}$, and $\{2\}$, so $|\mathcal{S}_2| = 3 = F_4$.

Now let $n \geq 2$, and suppose that for all $1 \leq k \leq n$ that $|\mathcal{S}_k| = F_{k+2}$. We will now count the subsets of $[n+1]$ which do not contain any consecutive integers. Let A be such a subset. We split into two cases: either A contains $n+1$, or it does not.

Case 1: $n + 1 \notin A$

In this case, A is actually a subset of $\{1, 2, \ldots, n\}$. It must not contain any two consecutive integers by assumption, so $A \in \mathcal{S}_n$. Moreover, any set in $\mathcal{S}_n$ will also belong to $\mathcal{S}_{n+1}$.

Thus there are $\mathcal{S}_n$-many sets that fall into this case.

Case 2: $n + 1 \in A$

In this case, let $A' = A \setminus \{n+1\}$ (every element of A except for $n+1$). A' still contains no consecutive integers. Since $n + 1 \in A$, we know $n \notin A$, since n and $n + 1$ are consecutive. This means that $A' \subset \{1, 2, \ldots n - 1\}$, so $A' \in \mathcal{S}_{n-1}$. Similarly to Case 1, any set in $\mathcal{S}_{n-1}$ will give a valid set A.

Thus there are $\mathcal{S}_{n-1}$-many sets that fall into this case.

Since these two cases are exhaustive, we conclude that $|\mathcal{S}_{n+1}| = |\mathcal{S}_n| + |\mathcal{S}_{n-1}|$. We know by induction that $|\mathcal{S}_n| = F_{n+2}$ and $|\mathcal{S}_{n-1}| = F_{n+1}$, so we have that

$$||\mathcal{S}_{n+1}| = F_{n+2} + F_{n+1} = F_{n+3} = F_{(n+1)+2}$$

which completes the induction.

Finally, this tells us that the answer to this question is F_{12}, or 144.

(j) This is actually somewhat related to part (i), though it doesn't immediately seem that way. Let's partition [12] into the following sets: $\{1, 2, 4, 8\}$, $\{3, 6, 12\}$, $\{5, 10\}$, $\{7\}$, $\{9\}$, and $\{11\}$. The key observation is that we only have to worry about interactions between numbers in the same partition set: 2 only affects our ability to include 1 and 4, but is entirely independent of whether we include 3 or not. Thus when we're building our set, our choice of which elements to include from each partition set are independent of each other, and independence is a wonderful thing in counting problems.

Let's look at $\{1, 2, 4, 8\}$. If we don't want to include any elements that are double another, this means we can't include a pair of elements which are consecutive in this list. But we know how many such sets there are - it's just $|\mathcal{S}_4|$ from (i), or $F_6 = 8$.

Similarly, we have 5 possible subsets of $\{3, 6, 12\}$, 3 from $\{5, 10\}$, and 2 each from the remaining 3 sets.

Since our choices from each partition set are independent, our total number of choices is $8 \times 5 \times 3 \times 2^3 = 960$.

□