Math 184 Homework 1 Solutions

Fall 2022

Question 1 (Ackermann Function, 50 points)**.** *The Ackermann function is a function $A(m, n)$ that takes as input two integers $m, n \geq 0$ and returns an integer. For all $n, m \geq 0$, it satisfies the recurrence relation:*

$$\begin{aligned} A(0, n) &= n + 1 \\ A(m + 1, 0) &= A(m, 1) \\ A(m + 1, n + 1) &= A(m, A(m + 1, n)). \end{aligned}$$

Prove that the above recurrence relations uniquely define the value of $A(m, n)$ for all integers $m, n \geq 0$. Hint: You will want to use induction, but you will need to be careful about what you are inducting on and how you set it up.

Proof. We begin with induction on m.

Base case: Suppose $m = 0$, then $A(0, n) = n + 1$ for all $n \geq 0$. This is a linear function in n, so $A(0, n)$ is uniquely defined for all $n \geq 0$.

Inductive step: Assume for $k > 0$ that $A(k, n)$ is uniquely defined for all $n \geq 0$. We want to prove that $A(k + 1, n)$ is uniquely defined for all $n \geq 0$. To do this, we use induction on n.

Base case: Suppose $n = 0$, then $A(k + 1, 0) = A(k, 1)$ by the second recurrence relation. The right hand side is well defined by the inductive hypothesis from induction on m.

Inductive step: Assume for $j > 0$ that $A(k+1, j)$ is uniquely defined. We want to prove that $A(k+1, j+1)$ is uniquely defined. Note by the third recurrence relation, this quantity is $A(k, A(k + 1, j))$. From the inductive hypothesis from induction on n, we know $A(k + 1, j)$ is uniquely defined as some integer l. Thus we have $A(k + 1, j + 1) = A(k, l)$, and by either the base case or the inductive hypothesis from induction on n, this is uniquely defined. This concludes our proof that $A(k + 1, n)$ is uniquely defined for all $n \geq 0$.

Now we conclude our inductive proof on m: We complete the inductive step by observing $A(k + 1, n)$ is uniquely defined for $n \geq 0$, so we have proved that $A(m, n)$ uniquely defined for all $m, n \geq 0$. □

Question 2 (Generalized Binary Representation, 50 points)**.** *Let $a_1 < a_2 < a_3, \ldots$ be a sequence of positive integers with $a_1 = 1$ and $a_n < a_{n+1} \leq 2a_n$. Show that any positive integer m can be written as a sum of distinct elements of this sequence.*

Proof. Fix such a sequence $a_1, a_2, \ldots$, and let $P(n)$ be the statement that n can be expressed as a sum of distinct entries of the sequence. We will prove that $P(n)$ holds for all n by strong induction on n.

Our base case is $n = 1$. In this case, we can write $1 = a_1$, since we know $a_1 = 1$.

Now suppose that $P(k)$ holds for all $1 \leq k < n$. We need to show that we can express n as a sum of distinct entries of the sequence. Let a_t be the largest entry of the sequence which is less than or equal to n. If $a_t = n$, then that is our expression[1]. Otherwise $a_t < n$. Let $m = n - a_t$. Then $1 \leq m < n$, so by the induction hypothesis we can write $m = a_{i_1} + a_{i_2} + \ldots + a_{i_l}$ as a sum of distinct entries of the sequence. Then we write $n = a_{i_1} + a_{i_2} + \ldots + a_{i_l} + a_t$. Now we need to check that these are all distinct. We already know that the a_is are distinct from the induction hypothesis, so all we need is to ensure that a_t is not among those.

We will show that $a_t > m$, which will imply that a_t cannot be included in the sum for m. Towards a contradiction, suppose that $m \geq a_t$. Then $n = m + a_t \geq 2a_t$. But by the rules of the sequence, we know that

[1] We can actually avoid this special case if we allow $n = 0$, and say that 0 is the sum of no elements of the sequence, using that as a second base case

$a_t < a_{t+1} \leq 2a_t$, so $a_t < a_{t+1} < n$, which contradicts the fact that a_t was the largest element of the sequence. Thus $a_t > m$, so a_t is distinct from all the $a_i s$ before it, which means that $n = a_{i_1} + a_{i_2} + \ldots + a_{i_l} + a_t$ is a way to express n as a sum of distinct elements of the sequence. This shows $P(n)$ holds, which completes the induction.

Hence we can express every natural number n as a sum of distinct elements of the sequence. $\square$