Math 184 Homework 6

Fall 2022

This homework is due on gradescope Friday November 11th at 11:59pm pacific time. Remember to justify your work even if the problem does not explicitly say so. Writing your solutions in LaTeXis recommend though not required.

Question 1 (Binomial Theorem Identity, 20 points)**.** *Notice that* $(1-x^2)^n = (1+x)^n(1-x)^n$*. By expanding both sides using the Binomial Theorem and comparing terms, derive a combinatorial identity.*

Question 2 (Generating Functions for Polynomials, 20 points)**.** *Show that for* m *a non-negative integer that*

$$\sum_{n=0}^{\infty} n^m x^n = \sum_{k=0}^{m} S(m,k)k! \left(\frac{x^k}{(1-x)^{k+1}}\right).$$

Question 3 (Finite Differences, 30 points)**.** *Given a sequence* $a(0), a(1), a(2), \ldots$*, define the first finite difference to be another sequence* $\Delta a(n) = a(n) - a(n-1)$ *for* $n \geq 1$*. Define the* k^{th} *finite difference* $\Delta^k a(n)$ *to be the sequence obtained by applying the finite difference operation to the sequence* $a(i)$ *a total of* k *times.*

(a) Given the generating function $A(x) = \sum_{n=0}^{\infty} a(n)x^n$ *give a formula for the generating function for its first finite difference* $B(x) = \sum_{n=1}^{\infty} \Delta a(n)x^n$*. [15 points]*

(b) Show that for the sequence $a(n) = n^k$ *for some integer* $k \geq 0$ *that the* $(k+1)^{st}$ *finite difference* $\Delta^{k+1}a(n)$ *is* 0 *for all sufficiently large* n*. Hint: You may want to use the result from Question 2 above. [15 points]*

Question 4 (Generating Function Calculations, points)**.** *Derive formulas for the following generating functions:*

(a) The sequence a_n *is given by the recurrence* $a_0 = a_1 = a_2 = 1$ *and* $a_n = a_{n-2} + a_{n-3}$ *for all* $n \geq 3$*. Give a formula for the generating function* $A(x) = \sum_{n=0}^{\infty} a_n x^n$*. [10 points]*

(b) The sequence b_n *is given by the recurrence* $b_0 = 1$ *and*

$$b_n = \begin{cases} b_{n/2} & \text{if } n \text{ is even} \\ -b_{(n-1)/2} & \text{if } n \text{ is odd} \end{cases}$$

for all $n \geq 0$*. Show that the generating function* $B(x) = \sum_{n=0}^{\infty} b_n x^n$ *satisfies* $B(x) = (1-x)B(x^2)$*. Give a formula for* $B(x)$ *as an infinite product. [10 points]*

(c) The sequence c_n *satisfies the recurrence relation* $c_n = \frac{2}{n}\sum_{m=0}^{n-1} c_m + n$*. Give a differential equation satisfied by the generating function* $C(x) = \sum_{n=0}^{\infty} c_n x^n$*. [10 points]*

Question 5 (Extra credit, 1 point)**.** *Approximately how much time did you spend working on this homework?*