Math 184 Homework 5

Fall 2022

This homework is due on gradescope Friday November 4th at 11:59pm pacific time. Remember to justify your work even if the problem does not explicitly say so. Writing your solutions in LaTeXis recommend though not required.

Question 1 (Odd Permutation Recursion, 30 points)**.** *Use a direct combinatorial proof to show that (using the convention that* $\mathrm{ODD}(0) = 1$*) for* $n \geq 1$

$$\mathrm{ODD}(n) = \sum_{\substack{1 \leq k \leq n \\ k \ odd}} (n-1)_{k-1} \mathrm{ODD}(n-k).$$

Note: You can use this recursion to provide an alternative proof of the formula for $\mathrm{ODD}(n)$*.*

Question 2 (Odd Number of Sets Inclusion Exclusion, 30 points)**.** *Let* $A_1, A_2, \ldots, A_n$ *be finite sets. Show that the number of elements* x *that are in an* odd *number of the* A_i*'s is given by*

$$\sum_{k=1}^{n} (-2)^{k-1} \sum_{1 \leq i_1 < i_2 < \ldots < i_k \leq n} |A_{i_1} \cap A_{i_2} \cap \cdots A_{i_k}|.$$

Hint: determine how many times each element x *is counted. You may want to use the Binomial Theorem.*

Question 3 (Necklace Counting, 40 points)**.** *You are trying to create a necklace with* n *beads labelled* $1, 2, \ldots, n$ *arranged in a circle. Each bead is one of* k *different colors and adjacent beads must be different colors. Prove that the number of ways to create such a necklace is*

$$(k-1)^n + (-1)^n (k-1).$$

Hint: Use inclusion-exclusion on the conditions that a given pair of adjacent beads are the same color. You may need to use the binomial theorem to simplify your answer.

Question 4 (Extra credit, 1 point)**.** *Approximately how much time did you spend working on this homework?*